A variational approach to probing extreme events in turbulent dynamical systems

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INTRODUCTION

Extreme events are ubiquitous in a wide range of dynamical systems, including turbulent fluid flows, nonlinear waves, large-scale networks, and biological systems. We propose a variational framework for probing conditions that trigger intermittent extreme events in high-dimensional nonlinear dynamical systems. We seek the triggers as the probabilistically feasible solutions of an appropriately constrained optimization problem, where the function to be maximized is a system observable exhibiting intermittent extreme bursts. The constraints are imposed to ensure the physical admissibility of the optimal solutions, that is, significant probability for their occurrence under the natural flow of the dynamical system. We apply the method to a body-forced incompressible Navier-Stokes equation, known as the Kolmogorov flow. We find that the intermittent bursts of the energy dissipation are independent of the external forcing and are instead caused by the spontaneous transfer of energy from large scales to the mean flow via nonlinear triad interactions. The global maximizer of the corresponding variational problem identifies the responsible triad, hence providing a precursor for the occurrence of extreme dissipation events. Specifically, monitoring the energy transfers within this triad allows us to develop a data-driven short-term predictor for the intermittent bursts of energy dissipation. We assess the performance of this predictor through direct numerical simulations.

RESULTS

Variational formulation of extreme events

Consider the general evolution equations

\[ \partial_t u = N(u) \]  

(1A)

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(1A)
Mechanisms underpinning the intermittent bursts of the observable. Al-

though it is unlikely that a generic trajectory of the system passes exactly through one of the maximizers, by continuity, any trajectory passing through a sufficiently small open neighborhood of the maximizer (that is, the instability regions of Fig. 1B) will result in a similar ob-

servable burst.

We emphasize that an optimization problem similar to Eqs. 2A and

2B has been pursued before in special contexts. The largest finite-
time Lyapunov exponent can be formulated as Eqs. 2A and 2B, where the

observable is the amplitude of infinitesimal perturbations after finite

R. The maximizer is the corresponding finite-time Lyapunov vector

(20, 21). In a similar context, Pringle and Kerswell (22) seek optimal

finite-amplitude perturbations that trigger transition to turbulence in

the pipe flow. They formulate the unknown optimal perturbation as

the solution of a constrained optimization problem similar to Eqs. 2A

and 2B, with the observable being the $L^2$ norm of the fluid velocity

field. Ayala and Protas (23–25) consider the finite-time singularity formation

for Navier-Stokes equations. They also use a variational method to seek

the initial conditions that could lead to finite-time singularities. In these

studies, the emphasis is given to the analysis of the most “unstable”

states, but the physical properties of the attractor are not taken into

account.

The standard approach for solving the PDE-constrained optimization

problem (Eqs. 2A and 2B) is an adjoint-based gradient iterative

method (26–28). This method is computationally very expensive be-

cause, at each iteration, the gradient direction needs to be evaluated

as the solution of an adjoint PDE. If the growth time scale $\tau$ is small

compared to the typical time scales of the observable, then it is reason-

able to replace the finite-time growth problem (Eqs. 2A and 2B) with its

instantaneous counterpart

$$
\sup_{u_0 \in X} \frac{d}{dt} \bigg|_{t=t_0} I(u(t))
$$

(3A)

where

$$
\left\{ \begin{array}{l}
u(t) \text{ satisfies (Eqs. 1A to 1C)} \\
C(u_0) = c_0
\end{array} \right.
$$

(3B)

Problems 3A and 3B seek initial states $u_0$ for which the instantan-

eous growth of the observable $I$ along the corresponding solution

$u(t)$ is maximal.

We point out that the large instantaneous derivatives of $I$ do not

necessarily imply a subsequent burst in the observable because the

growth may not always be sustained at later times along the trajectory

$u(t)$. As a result, the set of solutions to this instantaneous problem may

\[ K(u) = 0 \] (1B)

\[ u(\cdot, t_0) = u_0(\cdot) \] (1C)

where $u: \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$ belongs to an appropriate function space $X$ and
completely determines the state of the system. The initial condition
$X \ni u_0: \Omega \to \mathbb{R}^d$ is specified at the time $t_0$ and $\Omega \subset \mathbb{R}^d$ is an open
bounded domain. The differential operators $\mathcal{N}$ and $\mathcal{K}$ can be poten-
tially nonlinear. A wide range of physical models can be written as a set

of partial differential equations (PDEs), as in Eqs. 1A to 1C. For in-

stance, for incompressible fluid flows, Eq. 1A is the momentum equa-
tion and Eq. 1B is the incompressibility condition where $\mathcal{K}(u) = \nabla \cdot u$. For

simplicity, we will denote a trajectory of Eqs. 1A to 1C by $u(t)$.

Let $I: X \to \mathbb{R}$ denote an observable whose time series $I(u(t))$ along a

typical trajectory $u(t)$ exhibits intermittent bursts (see Fig. 1A). Drawing

upon the near-integrable case, we view the system as consisting of a

background chaotic attractor, which has small regions of instability

(see Fig. 1B). Once a trajectory reaches an instability region, it is mo-

mentarily repelled away from the background attractor, resulting in a

burst in the time series of the observable. Our goal here is to probe the

instability region(s) by using a combination of observed data from the

system and the governing equations of the system. We also re-

quire the instability regions to have a nonzero probability of occurrence

under the natural flow of the dynamical system. This constraint is of

particular importance because it excludes “exotic” states with extreme
growth of $I$ but with a negligible probability of being observed in prac-
tice [see the constraint $\mathcal{C}(u_0) = c_0$ in Eq. 2B].

We formulate this task as a constrained optimization problem. As-
sume that there is a typical time scale $\tau \in \mathbb{R}^+$ over which the bursts in the

observable $I$ develop (see Fig. 1A). We seek initial conditions $u_0$ whose

associated observable $I(u(t))$ attains a maximal growth within time $\tau$.

More precisely, we seek the solutions to the constrained optimization

problem

$$
\sup_{u_0 \in X} (I(u(t_0 + \tau)) - I(u(t_0)))
$$

(2A)

where

$$
\left\{ \begin{array}{l}
u(t) \text{ satisfies (Eqs. 1A to 1C)} \\
\mathcal{C}(u_0) = c_0
\end{array} \right.
$$

(2B)

where the optimization variable is the initial condition $u(t_0) = u_0$ of

systems 1A to 1C. The set of critical states is required to satisfy the

constraints in Eq. 2B to enforce two important properties. The first

property ensures that $u(t)$ obeys the governing Eqs. 1A to 1C as opposed
to being an arbitrary one-parameter family of functions. The second

property $\mathcal{C}(u_0) = c_0$, where $C: X \to \mathbb{R}^k$, is a codimension $k$ constraint.

This constraint is enforced to ensure the nonzero probability of occur-

rence, that is, states that are sufficiently close to the chaotic background

attractor. The set of probabilistically feasible states can be generally de-

scribed by exploiting basic physical properties of the chaotic attractor

such as average energy along different components of the state space

or the second-order statistics. The precise form of the constraint

$\mathcal{C}(u_0) = c_0$ is problem-dependent and will shortly be discussed in more
detail. We point out that more general inequality constraints of the form

$c_{\min} \leq \mathcal{C}(u_0) \leq c_{\max}$ may also be used. However, the treatment of these

inequality constraints is not discussed in this paper.

We expect the set of solutions to problems 2A and 2B to unravel the

mechanisms underpinning the intermittent bursts of the observable. Al-

though it is unlikely that a generic trajectory of the system passes exactly

through one of the maximizers, by continuity, any trajectory passing

through a sufficiently small open neighborhood of the maximizer

(that is, the instability regions of Fig. 1B) will result in a similar ob-

servable burst.
differ significantly from the finite-time problem (Eqs. 2A and 2B). Nonetheless, the solutions to the instantaneous problem can still be insightful. In addition, as we show below, these solutions can be obtained at a relatively low computational cost.

**Optimal solutions**

First, we derive an equivalent form of Problems 3A and 3B. Taking the time derivative of the time series $I(u(t))$ yields $(d/dt)I(u(t)) = dI(u; \partial_t u)$, where $dl(u; v) := \lim_{\epsilon \to 0} [I(u + \epsilon v) - I(u)]/\epsilon$ denotes the Gâteaux differential of $I$ at $u$ evaluated along $v$. Using Eq. 1A to substitute for $\partial_t u$, we obtain the following optimization problem, which is equivalent to problems 3A and 3B

$$
\sup_{u \in X} J(u) \\
\text{subject to} \\
\begin{cases}
K(u) = 0 \\
C(u) = c_0
\end{cases}
$$

where

$$
J(u) := dl(u; N(u))
$$

Note that the first constraint in Eq. 3B is simplified because we have already used Eq. 1A, and it only remains to enforce Eq. 1B. For notational simplicity, we omit the subscript from $u_0$.

If $f : X \to \mathbb{R}$ is a continuous map and the subset $S = \{u \in X : K(u) = 0, C(u) = c_0\}$ is compact in $X$, problems 4A and 4B have at least one solution. This follows from the fact that the image of a compact set under a continuous transformation is compact. Therefore, $f(S) \subset \mathbb{R}$ is compact, which implies that $f(S)$ is bounded and closed (29). Therefore, $f$ is bounded and attains its maximum (and minimum) on $S$. The uniqueness of the maximizer is not generally guaranteed. However, the set of maximizers (and minimizers) of $f$ is a compact subset of $S$ (30).

As we show in section S1, if $X$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_X$ and the operator $K$ is linear, then every solution of the optimization problem (Eqs. 4A and 4B) satisfies the set of Euler-Lagrange equations

$$
\dot{f}'(u) + K^*(u) + \sum_{i=1}^k \beta_i C_i'(u) = 0 \quad (6A)
$$

$$
K(u) = 0 \quad (6B)
$$

$$
C(u) = c_0 \quad (6C)
$$

Here, $K^*$ is the adjoint of $K$, and $f'(u)$ and $C_i'(u)$ are the unique identifiers of the Gâteaux differentials $dl(u; \cdot)$ and $dC_i(u; \cdot)$, such that $dl(u; v) = \langle f'(u), v \rangle_X$ and $dC_i(u; v) = \langle C_i'(u), v \rangle_X$ for all $v$. The existence and uniqueness of $f'(u)$ and $C_i'(u)$ are guaranteed by the Riesz representation theorem (31). Here, $C_i$ are the components of the map $C = (C_1, C_2, \cdots, C_k)$. The function $\alpha : \Omega \to \mathbb{R}$ and the vector $\beta = (\beta_1, \cdots, \beta_k) \in \mathbb{R}^k$ are unknown Lagrange multipliers to be determined together with the optimal state $u : \Omega \to \mathbb{R}^d$.

**Application to Navier-Stokes equations**

We consider the Navier-Stokes equations

$$
\partial_t u = -u \cdot \nabla u - \nabla p + \nu \Delta u + f, \quad \nabla \cdot u = 0
$$

where $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$ is the fluid velocity field, $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is the pressure field, and $\nu = Re^{-1}$ is the nondimensional viscosity, which coincides with the reciprocal of the Reynolds number $Re$. Here, we consider two-dimensional flows ($d = 2$) over the domain $\Omega = [0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions. The flow is driven by the monochromatic Kolmogorov forcing $f(x) = \sin(k_j y)e_1$, where $k_j = (0, k_j)$ is the forcing wave number and the vectors $e_i$ denote the standard basis in $\mathbb{R}^2$. In the following, we assume that the velocity fields are square integrable for all times, that is, $X = L^2(\Omega)$.

The kinetic energy $E$ (per unit volume), the energy dissipation rate $D$, and the energy input rate $I$ are defined, respectively, by

$$
E(u) = \frac{1}{|\Omega|} \int_\Omega \frac{1}{2} |u|^2 \, dx,
$$

$$
D(u) = \frac{\nu}{|\Omega|} \int_\Omega |\nabla u|^2 \, dx,
$$

$$
I(u) = \frac{1}{|\Omega|} \int_\Omega u \cdot f \, dx
$$

where $|\Omega|$ denotes the area of the domain, that is, $|\Omega| = (2\pi)^2$. Along any trajectory $u(t)$, these three quantities satisfy $E = I - D$. We use the energy dissipation rate $D$ to define the eddy turnover time, $\tau_e = \sqrt{E/D}$, where $E$ denotes the expected value.

The Kolmogorov flow admits the laminar solution $u = (Re/k_j^2) \sin(k_j y)e_1$. For the forcing wave number $k_j = 1$, the laminar solution is the global attractor of the system at any Reynolds number $Re$ (3). If the forcing applied is higher, the Reynolds number becomes sufficiently large, then the laminar solution becomes unstable. In particular, numerical evidence suggests that, for $k_j = 4$, and sufficiently large Reynolds numbers, the Kolmogorov flow is chaotic and exhibits intermittent bursts of energy dissipation (18, 33, 34). This is manifested in Fig. 2A, showing the time series of the energy dissipation $D$ at Reynolds number $Re = 40$ with $k_j = 4$.

A closer inspection reveals that each burst of the energy dissipation $D$ is shortly preceded by a burst in the energy input $I$ (see Fig. 2B). Therefore, we expect the mechanism behind the bursts in the energy input to be also responsible for the bursts in the energy dissipation. As we show in section S2.1, the energy input is given by $I(u) = -|a(k_j)\sin(\phi(k_j))|$, where $a(k)$ are the Fourier modes of the velocity field $u$ and $\phi(k)$ are their corresponding phases such that $a(k) = |a(k)| \exp(i\phi(k))$. We refer to the Fourier mode $a(k_j)$ as the mean flow. The energy input $I$ can grow through two mechanisms: (i) alignment between the phase of the mean flow and the external forcing, that is, $\phi(k_j) \to -\pi/2$, and (ii) growth of the mean flow energy $|a(k_j)|^2$.

Examining the alignment between the forcing and the velocity field rules out mechanism (i) (cf. fig. S1). The remaining mechanism (ii) is possible through the nonlinear term in the Navier-Stokes equation. This nonlinearity redistributes the system energy among various Fourier modes $a(k)$ through triad interactions of the modes whose wave numbers $(k, p, q)$ satisfy $k = p + q$ (see section S2.2 for further details). Because of the high number of active modes involved in the intricate network of triad interactions, it is unclear which triad (or triads) is (are) responsible for the nonlinear transfer of energy to the mean flow during the extreme events. As we show below, our variational approach
Excluding the intermittent bursts, the energy dissipation of the Kolmogorov flow exhibits small oscillations around its mean value $\mathbb{E}[D]$ (see Fig. 2A). On the basis of this observation, we seek optimal solutions of Eqs. 4A and 4B, which are constrained to have the energy dissipation $D = \mathbb{E}[D]$. This results into the constraint (Eq. 9) with $A = \nabla$ and $C(u) = c_0 = \mathbb{E}[D] \times (Re/2)$. We approximate the mean value $\mathbb{E}[D]$ from direct numerical simulations. At $Re = 40$, for instance, we have $\mathbb{E}[D] = 0.117$.

**Probing the extreme energy transfers**

The functional $J$ (see Eq. 5) associated with the energy input $I$ reads

$$J(u) = \frac{1}{|\Omega|} \int_\Omega [u \cdot (u \cdot \nabla f) + \nu u \cdot (\Delta f)] \, dx$$  \hfill (10)

The associated Euler-Lagrange equations (Eqs. 6A to 6C) read

$$\begin{align*}
(\nabla f + \nabla f^T)u + \nu \Delta f - \nabla \alpha + \beta A^T A u & = 0 \quad (11A) \\
\nabla \cdot u & = 0 \quad (11B) \\
\frac{1}{|\Omega|} \int_\Omega |A(u)|^2 \, dx & = c_0 \quad (11C)
\end{align*}$$

where $J'(u) = (\nabla f + \nabla f^T) u + \nu \Delta f$, $K^0(\alpha) = -\nabla \alpha$, and $C'(u) = A^T A u$ (see section S2.3 for the derivations). We set $A = \nabla$ to enforce a constant energy dissipation constraint. This implies that $A^T A u = -\Delta u$.

Using the symmetries of Eqs. 11A to 11C, we find that it admits the pair of exact solutions $u_\pm = \pm (2\sqrt{c_0}/k_f) \sin(k_f y)e_1$, $\alpha_0 = \pm (2\sqrt{c_0}/k_f) \sin(2k_f y)dy$, and $\beta_0 = \pm (2\sqrt{c_0}/k_f)$. More complex solutions, with unknown closed forms, may exist. We approximate these solutions using the Newton iterations described in section S3.

At each $Re$, we initiated several Newton iterations from random initial conditions. In addition to the pair of exact solutions $(u_\pm, \alpha_0, \beta_0)$, the iterations yielded one nontrivial solution. Figure 3 shows the resulting three branches of solutions including the exact solution $u_\pm$ (solid black), the exact solution $u_0$ (dashed black), and the nontrivial solution (red circles). For small Reynolds numbers, our Newton searches only returned the exact solutions. At $Re = 3.1$, a bifurcation takes place where a new nontrivial solution is born. This solution appears to be a global maximizer because no other solutions were found. Because the intermittent bursts are only observed for $Re > 35$, we focus the following analysis on this range of Reynolds numbers.

The nontrivial optimal solution converges to an asymptotic limit as the $Re$ increases. This is discernible from the plateau of the red curve in Fig. 3 and the select solutions shown in its outset. The three most dominant Fourier modes present in this asymptotic solution are the forcing wave number $(0, k_f)$ and the wave numbers $(1, 0)$ and $(1, k_f)$ together with their complex conjugate pairs. Incidentally, these wave numbers form a triad, $(0, k_f) + (1, 0) = (1, k_f)$. The dominant mode of the optimal solution corresponds to the wave number $(1, 0)$ whose modulus $|a(1, 0)|$ is one order of magnitude larger than the other nonzero modes.

Next, we turn to the direct numerical simulations of the Kolmogorov flow and monitor the three Fourier modes $a(0, k_f), a(1, 0)$, and $a(1, k_f)$. We find that the energy transfers within this triad underpin the intermittent
observing the modulus $|a(1, 0, t)|$. More specifically, relatively small
values of $\lambda(t) := |a(1, 0, t)|$, along a solution $u(t)$, signal the high probability
of an upcoming burst in the energy dissipation.

To quantify this, we consider the conditional probability

$$
\mathcal{P}(D_1 \leq D_m(t) \leq D_2 | \lambda_1 \leq \lambda(t) \leq \lambda_2)
$$

(12)

where $D_m(t) = \max_{\tau \in [t + t_i + t_f]} |D(u(\tau))|$ is the maximum of the energy dissipation over the future time interval $[t + t_i, t + t_f]$. This conditional probability measures the likelihood of the future maximum value of the energy dissipation belonging to the interval $[D_1, D_2]$, given that the present value of the indicator $\lambda(t)$ belongs to $[\lambda_1, \lambda_2]$. The constant parameters $t_i > t_f > 0$ determine the future time interval $[t + t_i, t + t_f]$. The length of the time window $t_f - t_i$ is long enough to ensure that the extreme event (if it exists) is contained in the time interval $[t + t_i, t + t_f]$. The choice of the prediction time $t_i$ will be discussed shortly. The reported results are robust to small perturbations to all parameters.

Figure 5A shows the conditional probability density corresponding to Eq. 12. We observe that relatively small values of $\lambda$ correlate strongly with the high future values of the energy dissipation $D$. For instance, when $\lambda < 0.4$, the value of $D_m$ is most likely larger than 0.2. Conversely, when $\lambda$ is larger than 0.4, the future values of the energy dissipation $D_m$ are smaller than 0.2.

We seek an appropriate value $\lambda_0$ such that $\lambda(t) < \lambda_0$ predicts an extreme burst of energy dissipation over the future time interval $[t + t_i, t + t_f]$. Denote the extreme event threshold by $D_e$, such that $D > D_e$ constitutes an extreme burst of energy dissipation. We define the probability of an upcoming extreme event $P_{ee}$ as

$$
P_{ee}(\lambda_0) = \mathcal{P}(D_m(t) > D_e | \lambda(t) = \lambda_0)
$$

(13)

which measures the likelihood that $D_m(t) > D_e$ assuming that $\lambda(t) = \lambda_0$. Here, we set the threshold of the extreme event $D_e$, as the mean value of the energy dissipation plus 2 SDs, $D_e = \mathbb{E}[D] + 2 \sqrt{\mathbb{E}[D^2] - \mathbb{E}[D]^2} = 0.194$.

The extreme event probability $P_{ee}$ can be computed from the probability density shown in Fig. 5A (see section S5 for details).

Figure 5B shows the probability of extreme events $P_{ee}$ as a function of the parameter $\lambda_0$. If, at time $t$, the values of $\lambda(t)$ are larger than 0.5, the probability of a future extreme event, that is, $D_m(t) > D_e$, is nearly zero. The probability of a future extreme event increases as $\lambda(t)$ decreases. At $\lambda(t) \approx 0.4$, the probability is 50%. If $\lambda(t) < 0.3$, then the likelihood of an upcoming extreme event is nearly 100%. The horizontal dashed line in Fig. 5A marks the transition line from the low likelihood of an upcoming extreme event $P_{ee} < 0.5$ to the higher likelihood $P_{ee} > 0.5$. This line, together with the vertical line $D_m = D_e$ divides the conditional probability density into four regions: (I) correct rejections $P_{ee} < 0.5$ and $D_m(t) < D_e$; (II) correct prediction of no upcoming extremes; (II) false positives $P_{ee} > 0.5$ but $D_m(t) < D_e$; the indicator predicts an upcoming extreme event but no extreme event actually takes place; (III) hits $P_{ee} > 0.5$ and $D_m(t) > D_e$; correct prediction of an upcoming extreme event; and (IV) false negatives $P_{ee} < 0.5$ but $D_m(t) > D_e$; an extreme event takes place, but the indicator fails to predict it.

A reliable indicator of upcoming extreme events must maximize the number of correct rejections (quadrant I) and hits (quadrant III) while, at the same time, having minimal false positives (quadrant II) and false negatives (quadrant IV). From nearly 100,000 predictions made, only 0.26% false negatives and 0.85% false positives were recorded. The number
of hits was 5.6%, and the number of correct rejections amounts to 93.3% of all predictions made. As we show in the Supplementary Materials, this amounts to a 95.6% success rate for the prediction of the extreme events (see eq. S24 and table S1). Note that the high percentage of correct rejections compared to the hits is a mere consequence of the fact that the extreme events are rare.

An additional desirable property of an indicator is its ability to predict the upcoming extremes well in advance of the events taking place. The chosen prediction time $t_i = 1$ is approximately twice the eddy turnover time $t_e$. In comparison, it takes approximately one eddy turnover time (on average) for the energy dissipation rate to grow from $E[D] + 2\sqrt{E[|D|^2] - E[D]^2}$ to its extreme value $D_e = E[D] + 2\sqrt{E[|D|^2] - E[D]^2}$.

The prediction time $t_i$ can always be increased at the cost of increasing false positives and/or false negatives. For instance, with the choice $t_i = 2 \approx 4.3t_e$ and $t_f = 3 \approx t_i + 2.2t_e$, prediction of the extreme events $D_m > D_e$ returns 1.2% false negatives and 0.6% false positives. The number of hits decreases slightly to 5.3%, as does the number of correct rejections (92.9%), which amounts to a success rate of 82% in the extreme event prediction (see eq. S24). Therefore, the prediction time $t_i = 2$ still yields satisfactory predictions. Upon increasing $t_i$ further, eventually, the number of hits becomes comparable to the number of false negatives, at which point the predictions are unreliable.

**DISCUSSION**

A method for the computation of precursors of extreme events in complex turbulent systems is introduced here. The new approach combines basic physical properties of the chaotic attractor (such as energy distribution along different directions of phase space) obtained from data with stability properties induced by the governing equations. The method is formulated as a constrained optimization problem, which can be solved explicitly if the time scale of the extreme events is short compared to the typical time scales of the system. To demonstrate the approach, we consider a stringent test case, the Kolmogorov flow, which has a turbulent attractor with positive Lyapunov exponents and intermittent extreme bursts of energy dissipation. We can correctly identify the triad of modes associated with the extreme events. Moreover, the derived precursors allow for the formulation of an accurate short-term prediction scheme for the intermittent bursts. The results demonstrate the robustness and applicability of the approach on systems with high-dimensional chaotic attractors.
MATERIALS AND METHODS

The Navier-Stokes equations and the corresponding Euler-Lagrange equations were solved numerically with a standard pseudospectral code with $N \times N$ Fourier modes and 2/3 dealiasing. For $Re = 60$, 80, and 100, we used $N = 256$ to fully resolve the velocity fields. However, at $Re = 40$, this resolution was unnecessarily high, and hence, we used $N = 128$. The temporal integration of the Navier-Stokes equations was carried out with a fourth-order Runge-Kutta scheme.

SUPPLEMENTARY MATERIALS

Supplementary material for this article is available at http://advances.sciencemag.org/cgi/content/full/3/9/e1701533/DC1

REFERENCES AND NOTES


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