Deck effects on the statistical structure of the vertical bending moment loads during random waves: an analytical approach

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ABSTRACT

The effects of a finite deck on the statistical distribution of the vertical bending moment (VBM) for ships subjected to random waves are presented. In previous work, analytical approximations were obtained for the probability density function (PDF) of the VBM by expanding the Froude-Krylov forces around the undisturbed, pitch-inclined water plane in terms of the wave amplitude, where the small parameter is the wave slope. By maintaining up to second order terms, the heavy-tailed structure of the PDF is reconstructed. However, this is eventually tamed by finite-deck effects, since deck submergence eventually results in slower VBM growth. To analyze this effect, the correlation structure is derived between the random wave amplitudes and the induced pitch motion. This is done analytically given the Gaussian structure of the pitch dynamics. Subsequently, a Bayesian argument is employed to obtain the conditional statistics for the waves associated with pitch angles that do not cause deck submergence. Using this non-Gaussian distribution for random waves, the resulting PDF tail for the VBM has an inflection point, which interrupts the heavy-tailed structure. Analytical findings are compared with direct numerical simulations by the Large Amplitude Motion Program (LAMP).

INTRODUCTION

The effect of a finite deck on the extreme value properties of the vertical bending moment (VBM) for ships subjected to random waves is presented. In previous work, analytical approximations were obtained for the probability density function (PDF) of the VBM by expanding the Froude-Krylov forces around the undisturbed, pitch-inclined water plane in terms of the wave amplitude, where the small parameter is the wave slope. By maintaining up to second order terms, the heavy-tailed structure of the PDF is reconstructed. However, this is eventually tamed by finite-deck effects, since deck submergence eventually results in slower VBM growth. To analyze this effect, the correlation structure is derived between the random wave amplitudes and the induced pitch motion. This is done analytically given the Gaussian structure of the pitch dynamics. Subsequently, a Bayesian argument is employed to obtain the conditional statistics for the waves associated with pitch angles that do not cause deck submergence. Using this non-Gaussian distribution for random waves, the resulting PDF tail for the VBM has an inflection point, which interrupts the heavy-tailed structure. Analytical findings are compared with direct numerical simulations by the Large Amplitude Motion Program (LAMP).

The subject of statistical responses for ship motions and loads was included in Belenky et al. (2018), which describes the application of physics-informed statistical extrapolation of roll motion data to predict the probability of extreme roll angles and capsizing. A single degree-of-freedom (DOF) dynamical system with piecewise linear stiffness was a reduced-order mathematical model for roll motion and capsizing. The model is qualitatively correct, as it reproduces the topology of the phase plane. Piecewise linear stiffness allows the derivation of a closed-form expression for the peaks of roll motion. Then, the distribution of roll peaks can also be derived in analytical form. The tail of the roll peak distribution has a complex structure: a heavy tail from the vicinity of the maximum of the roll restoring (GZ) curve, which then becomes light and has an upper bound near the angle of vanishing stability (Belenky et al., 2019). A data perspective of this approach for VBM is given by Brown and Pipiras (2020).

CORRELATION BETWEEN WAVE AMPLITUDES AND PITCH MOTION

An understanding of the relationship between the random wave excitation and the induced pitch motions is the first
step of the analysis. To quantify the probability of deck submergence is an essential step. Direct simulations with LAMP on the ONR Topsides Series Flared hull were performed with the configurations described in Sapsis et al. (2020) (head seas with ship speed 10 knots and significant wave height 7.62 m).

In LAMP there are several ways to treat deck submergence. The most straightforward and most commonly used is a ‘deck-in-water’ approach in which the body-nonlinear Froude-Krylov plus Hydrostatic pressure is applied directly to the deck surface. This is, implicitly, the scheme used in the volume based calculation (SimpleCode). The best way is to compute the water-on-deck-flow using a shallow-water finite-volume calculation (SimpleCode). The best way is to compute the water-on-deck-flow using a shallow-water finite-volume method, but this model is very expensive and difficult to run.

These simulations confirm that the pitch angle follows closely a normal distribution, Fig. 1. A single wavenumber stochastic approximation is applied for the random sea with length scale equal to the ship length:

\[
h(x,t; \zeta) = a_c(t; \zeta) \cos \left( \frac{2 \pi x}{L} \right) + a_s(t; \zeta) \sin \left( \frac{2 \pi x}{L} \right)
\]

where \(a_c, a_s\) are uncorrelated stochastic processes with given spectrum, while \(L\) is the ship length. The argument \(\zeta\) denotes the random realization. A linear model for pitch dynamics was motivated by the Gaussian pitch response. While the present study only considers head seas, a quick check of calculations (for other hulls) shows pitch to be fairly well fitted by a Gaussian distribution for following sea cases as well.

![Figure 1: Statistics of pitch angle directly computed from LAMP superimposed with an optimal Gaussian fit.](image-url)

The area below the water surface, at each location \(x\) of the ship is

\[
A(x, \theta x + h(x,t)),
\]

where \(\theta\) is the pitch angle, \(x\) is measured with respect to the midship section (positive towards the bow), and \(h(x,t)\) is the wave elevation. The function \(A\) is monotonically increasing; therefore, invertible with respect to its second argument. The moment that gets into the pitch equation is the Froude-Krylov plus Hydrostatic moment

\[
M_{FKHS}(\theta; h(x,t; \zeta)) = \int_{-L/2}^{L/2} x A(x, \theta x + h(x,t; \zeta)) dx.
\]

A Taylor series expands the function \(A\) around the undisturbed pitch-inclined water plane \(\theta x\), (without imposing any constraint on the magnitude of \(h\)) results in:

\[
A(x, \theta x + h) = A(x, \theta x) + \sum_{q=1}^{Q} \frac{A^{(q)}(x, \theta x)}{q!} h^q
\]

where \(Q\) is chosen according to the magnitude of \(h\) and how complicated the shape of \(A(x, d)\) is. Based on this:

\[
M_{FKHS}(\theta, t; \zeta) = \int_{-L/2}^{L/2} x A(x, \theta x) dx
\]

\[
+ \sum_{q=1}^{Q} \int_{-L/2}^{L/2} x A^{(q)}(x, \theta x) \frac{h^q(x,t; \zeta)}{q!} dx.
\]

The first term is the hydrostatic pitch-restoring moment, while the second term expresses the interaction with the random waves.

The last expansion is decomposed as:

\[
M_{FKHS}(\theta, t; \zeta) = M_{HS}(\theta) + M_{FK}(\theta, t; \zeta),
\]

where for a fixed \(\theta\) the first term is deterministic, given by the hydrostatic restoring moment:

\[
M_{HS}(\theta) = \int_{-L/2}^{L/2} x A(x, \theta x) dx,
\]

and the term \(M_{FK}\) consists of the first- and second-order terms:

\[
M_{FK}(\theta, t; \zeta) = \int_{-L/2}^{L/2} x A'(x, \theta x) h(x,t; \zeta) dx + \frac{1}{2} \int_{-L/2}^{L/2} x A''(x, \theta x) h^2(x,t; \zeta) dx.
\]

The first order terms have Gaussian statistics (for fixed \(\theta\)) and are given by

\[
\int_{-L/2}^{L/2} x A'(x, \theta x) h(x,t; \zeta) dx = a_c(t; \zeta) \chi_c(\theta) + a_s(t; \zeta) \chi_s(\theta)
\]

where

\[
\chi_c(\theta) = \int_{-L/2}^{L/2} x A'(x, \theta x) \cos \left( \frac{2 \pi x}{L} \right) dx,
\]

\[
\chi_s(\theta) = \int_{-L/2}^{L/2} x A'(x, \theta x) \sin \left( \frac{2 \pi x}{L} \right) dx.
\]
These two functions are shown in Fig. 2. Given the Gaussian character of the response, the linear terms are kept (i.e. second order terms or higher are neglected). The hydrostatic moment is linearized. In this way, a linear model is obtained:

\[ I \ddot{\theta} + c \dot{\theta} + k \theta = a_c(t; \xi) \chi_c(\theta) + a_s(t; \xi) \chi_s(\theta) \]  

(11)

where

\[ H_{\theta a_c}(\omega) = \frac{\bar{\chi}_c}{k - I \omega^2 + ic \omega} \]

\[ H_{\theta a_s}(\omega) = \frac{\bar{\chi}_s}{k - I \omega^2 + ic \omega} \]

and \( S_{a_c a_s}(\omega) = S_{a_s a_c}(\omega) \triangleq S_0(\omega) \) is the power spectral density for the stochastic wave amplitudes. The cross-covariance terms are given by:

\[ C_{\theta a_c} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\chi}_c S_0(\omega) d\omega = \bar{\chi}_c \mathcal{F} \]

\[ C_{\theta a_s} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\chi}_s S_0(\omega) d\omega = \bar{\chi}_s \mathcal{F} \]

(14)

(15)

with \( \mathcal{F} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega) d\omega \). For the case of white noise excitation, \( S_0(\omega) = s_0 \),

\[ \mathcal{F} = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_0 d\omega = 0, \]

(16)

(Appendix A).

**Correlation coefficients from pitch statistics**

The moment equation obtained from the pitch model equation is applied. From ergodicity:

\[ \mathbb{E}[\ddot{\theta}] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \ddot{\theta} dt = -\sigma_\theta^2. \]

Similarly:

\[ \mathbb{E}[\dot{\theta}] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\theta} dt = 0. \]

Multiplying the model equation by \( \theta \) and averaging gives:

\[ I \ddot{\theta} + c \dot{\theta} + k \theta^2 = \bar{\chi}_c \theta a_c + \bar{\chi}_s \theta a_s. \]

Therefore,

\[ -I \sigma_\theta^2 + k \sigma_\theta^2 = \bar{\chi}_c C_{\theta a_c} + \bar{\chi}_s C_{\theta a_s}. \]

(17)

Combining this with the equations for \( C_{\theta a_c} \) yields:

\[ \mathcal{F} = \frac{k \sigma_\theta^2 - I \sigma_\theta^2}{\bar{\chi}_c + \bar{\chi}_s} \]

(18)

Hence, to tune the model one only needs the variance of the pitch angle and pitch velocity (not the full spectrum for the wave coefficients). Having obtained the correlation structure between the pitch angle and the wave coefficients, the statistical analysis of the waves that cause deck submergence follows.
VERTICAL BENDING MOMENT (VBM) AND CRITICAL PITCH ANGLE

Approximation of VBM

In previous work (Sapsis et al., 2020), good approximation properties were demonstrated for a quadratic expansion for the vertical bending moment with respect to the wave height. Specifically, referring to Fig. 3, the following expression is for the Froude-Krylov induced VBM at an arbitrary location $\psi$, i.e.:

$$
M_{FK-VBM}(\theta, h, \psi) = \int_{\psi}^{L/2} xA(x, \theta x) dx + \frac{1}{2} \int_{\psi}^{L/2} xA''(x, \theta x) h^2(x, t) dx + O(h^3).
$$

(19)

Figure 3: Derivation of the vertical bending moment at an arbitrary location along the length of the ship, $\psi$.

The above expansion does not include inertia terms and terms induced by damping effects. For what follows, the case of VBM at the mid-ship section $\psi = 0$ is considered. Next, details are given for the effects of wave-induced VBM:

$$
M_{W-VBM}(\theta, h) = \int_{0}^{L/2} xA'(x, \theta x) h(x, t) dx + \frac{1}{2} \int_{0}^{L/2} xA''(x, \theta x) h^2(x, t) dx + O(h^3).
$$

(20)

Utilizing the wave expression (11), the wave induced VBM approximation is

$$
M_{W-VBM}(\alpha_c, \alpha_s) = \rho_c(\theta) \alpha_c(t) + \rho_s(\theta) \alpha_s(t) + \rho_c^{2}(\theta) \alpha_c^2(t) + \rho_s^2(\theta) \alpha_s(t) \alpha_s(t).
$$

(21)

where,

$$
\rho_c(\theta) = \frac{1}{2} \int_{0}^{L/2} xA''(x, \theta x) \cos \left( \frac{2\pi x}{L} \right) dx,
$$

(22)

$$
\rho_s(\theta) = \frac{1}{2} \int_{0}^{L/2} xA''(x, \theta x) \sin \left( \frac{2\pi x}{L} \right) dx.
$$

(23)

These coefficients are presented in Fig. 4 as functions of the pitch angle $\theta$. The coefficients $\rho$ refer to the mid-ship section and they are different from the coefficients $\chi$, which involve integration over the full ship length.

Figure 4: Hydrodynamic coefficients $\rho(\theta)$ (involving the mid-ship section) for the ONR Topsides Fared hull.

As for the case of the pitch equation, derived previously, the pitch-averaged values of the hydrodynamic coefficients (eq. (12)) are considered. The numerical values are given in the table below:

<table>
<thead>
<tr>
<th>$\rho_c(\theta)$</th>
<th>$\rho_s(\theta)$</th>
<th>$\rho_c^{2}(\theta)$</th>
<th>$\rho_s^2(\theta)$</th>
<th>$\rho_s(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.183 \times 10^4$</td>
<td>$2.0260 \times 10^5$</td>
<td>$4.27 \times 10^3$</td>
<td>$6.23 \times 10^3$</td>
<td>$-3.76 \times 10^3$</td>
</tr>
</tbody>
</table>

In Fig. 5 a color plot is presented of the function $M_{W-VBM}(\alpha_c, \alpha_s)$ for the hull considered. The coefficients of the linear terms are much smaller compared with those of the nonlinear terms.
Figure 5: Analytical approximation of the wave-induced VBM (in kg-m), \( M_{W-VBM} \), with respect to the wave parameters, eq. (21).

Results for the induced PDF are shown in Fig. 6. While the analytical approximation does not include the effects of inertia and hydrostatics terms, it compares favorably with the full VBM computed by LAMP. In this comparison, both distributions are normalized by their respective standard deviations. This is the case until \( 4 - 5\sigma \), when the effect of deck submergence results in faster decay of the tail.

Critical pitch angle

The critical pitch angle, above which the effect of wave-induced VBM is very small, is computed. An obviously important angle for this case is the angle \( \theta_d \), when the deck enters the water (Fig. 7). Beyond this point, the part of the hull that is fully submerged does not contribute to the wave-induced VBM. The VBM due to the partially submerged hull is not modelled. Instead, the critical angle is approximated, above which the resulting wave-induced VBM is an order of magnitude smaller compared with the one when the hull is not submerged.

The VBM for pitch angles smaller than the critical one will be approximated with the full ship length, while the VBM for pitch larger than the critical angle is not included in the computed PDF.

Figure 7: Computation of the effective VBM for the case of deck submergence.

Specifically, when the deck is partially submerged the hydrodynamic coefficients become (Fig. 7)

\[
\rho_{\text{eff}}(\theta) = \frac{1}{k!} \int_0^{\tan \theta} x \frac{\partial^{k+l} A(x, z)}{\partial x^{k+l}} \left|_{z=\theta} \right. \times \\
\cos \left( \frac{2\pi x}{L} \right) \sin \left( \frac{2\pi x}{L} \right) dx, \quad \theta \geq \theta_d
\]  

(27)

since the part of the hull in the water does not contribute to wave-induced VBM (shaded region in Fig. 7). These integrals are approximated by assuming that the variation of the quantity under the integral does not vary significantly over the length of the ship:

\[
\rho_{\text{eff}}(\theta) = \tan \theta_d \rho_{\text{eff}}(\theta_d), \quad \theta \geq \theta_d.
\]  

(28)

Based on this approximation, when the deck has been submerged, the wave induced VBM is calculated as:

\[
M_{W-VBM}(\theta) = \frac{\tan \theta}{\tan \theta_d} M_{W-VBM}(\alpha_c, \alpha_s), \quad \theta \geq \theta_d
\]  

(29)

The critical angle is defined as the one for which

\[
M_{W-VBM}(\theta_{cr}) = q M_{W-VBM}(\alpha_c, \alpha_s),
\]  

(30)

where \( 0 < q < 1 \) is a parameter defining how much smaller the VBM is because of deck submergence (typically chosen around 0.4 – 0.5). Therefore, for small angles the critical angle is:

\[
\theta_{cr} = \frac{\theta_d}{q}.
\]  

(31)

For the considered hull, \( \theta_d = 5.87 \text{ deg} \), while the resulting critical angle is considered to be \( \theta_{cr} = 13.5 \text{ deg} \).

CONDITIONAL WAVE STATISTICS FOR DECK SUBMERGENCE

Now that the critical wave angle and the correlation structure between the pitch angle and the wave amplitudes have been determined, the conditional PDF is computed for \( \alpha_c, \alpha_s \), for which \( \theta \geq \theta_{cr} \). First, the conditional mean and covariance are computed.
Conditional mean and covariance of $\alpha_c, \alpha_s$ for $\theta \geq \theta_{cr}$

The PDF for the wave amplitudes $\hat{\alpha} = (\alpha_c, \alpha_s)$ is assumed to be Gaussian

$$f(\alpha_c, \alpha_s) = \frac{1}{2\pi \sigma_\alpha^2} \exp \left( -\frac{\alpha_c^2 + \alpha_s^2}{2\sigma_\alpha^2} \right). \quad (32)$$

From direct computation

$$\mu_{\hat{\alpha}|\theta \geq \theta_{cr}} = \int \hat{\alpha} f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) d\alpha_c d\alpha_s$$

$$= \int \hat{\alpha} \frac{f(\alpha_c, \alpha_s, \theta)}{P(\theta \leq \theta_{cr})} d\alpha_c d\alpha_s$$

$$= \int_{-\infty}^{\theta_{cr}} \frac{1}{P(\theta \leq \theta_{cr})} \int_{-\infty}^{\alpha} \frac{f(\alpha_c, \alpha_s, \theta)}{f(\theta)} f(\theta) d\alpha_c d\alpha_s d\theta$$

$$= \int_{-\infty}^{\theta_{cr}} \mu_{\hat{\alpha}|\theta = \theta^*} \frac{f(\theta^*)}{P(\theta \leq \theta_{cr})} d\theta^*$$

(33)

Since $\alpha_c, \alpha_s, \theta$ are jointly Gaussian, the conditional mean $\mu_{\hat{\alpha}|\theta = \theta^*}$ is given by

$$\mu_{\hat{\alpha}|\theta = \theta^*} = \begin{pmatrix} C_{\theta \hat{\alpha}_c} \\ C_{\theta \hat{\alpha}_s} \end{pmatrix} \frac{\theta^*}{\sigma_\theta^2}. \quad (34)$$

Combining the last two equations,

$$\mu_{\hat{\alpha}|\theta \leq \theta_{cr}} = \begin{pmatrix} C_{\theta \hat{\alpha}_c} \\ C_{\theta \hat{\alpha}_s} \end{pmatrix} \frac{\mu_{\theta|\theta \leq \theta_{cr}}}{\sigma_\theta^2}, \quad (35)$$

where $\mu_{\theta|\theta \leq \theta_{cr}}$ is the conditional mean for the pitch angle given that $\theta \leq \theta_{cr}$.

From a similar argument for the conditional covariance:

$$C_{\hat{\alpha} \hat{\alpha}|\theta \leq \theta_{cr}} = \int (\hat{\alpha} - \mu_{\hat{\alpha}|\theta \leq \theta_{cr}})(\hat{\alpha} - \mu_{\hat{\alpha}|\theta \leq \theta_{cr}})^T f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) d\alpha_c d\alpha_s$$

$$= \int (\hat{\alpha} - \mu_{\hat{\alpha}|\theta \leq \theta_{cr}})(\hat{\alpha} - \mu_{\hat{\alpha}|\theta \leq \theta_{cr}})^T \frac{f(\alpha_c, \alpha_s, \theta)}{P(\theta \leq \theta_{cr})} d\alpha_c d\alpha_s$$

$$= \int_{-\infty}^{\theta_{cr}} C_{\hat{\alpha} \hat{\alpha}|\theta = \theta^*} \frac{f(\theta^*)}{P(\theta \leq \theta_{cr})} d\theta^* = C_{\hat{\alpha} \hat{\alpha}|\theta = \theta^*},$$

where the last equality follows from the fact that the conditional covariance is independent of $\theta^*$ given by

$$C_{\hat{\alpha} \hat{\alpha}|\theta = \theta^*} = C_{\hat{\alpha} \hat{\alpha}} - \frac{1}{\sigma_\theta^2} \begin{pmatrix} C_{\theta \hat{\alpha}_c} \\ C_{\theta \hat{\alpha}_s} \end{pmatrix} \begin{pmatrix} C_{\theta \hat{\alpha}_c} \\ C_{\theta \hat{\alpha}_s} \end{pmatrix}^T \quad (36)$$

Figure 8: a) Conditional probability for pitch below the critical value for each wave amplitude; b) Conditional PDF for wave amplitudes $(\alpha_c, \alpha_s)$ given that the resulted pitch angle is less than critical, i.e. no deck submergence (solid blue). The dashed lines indicate contours of the unconditional PDF for the wave amplitudes.

Next, the full calculation of the conditional PDF is performed for the waves that cause pitch angles less than the critical one. From Bayes’ rule:

$$f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) = \frac{P(\theta \leq \theta_{cr}|\alpha_c, \alpha_s) f(\alpha_c, \alpha_s)}{P(\theta \leq \theta_{cr})}.$$

Now,

$$P(\theta \leq \theta_{cr}|\alpha_c, \alpha_s) = \int_{-\infty}^{\theta_{cr}} f(\theta|\alpha_c, \alpha_s) d\theta,$$
where \( f(\theta|\alpha_c, \alpha_s) \) follows a Gaussian distribution with mean:

\[
\hat{\theta}_{\alpha_c, \alpha_s} = \frac{C_{\theta\alpha_c}}{\sigma_\theta^2} \alpha_c + \frac{C_{\theta\alpha_s}}{\sigma_\theta^2} \alpha_s,
\]

and variance:

\[
\sigma^2_{\hat{\theta}|\alpha_c, \alpha_s} = \sigma_\theta^2 - \frac{C_{\theta\alpha_c}^2}{\sigma_\theta^4} - \frac{C_{\theta\alpha_s}^2}{\sigma_\theta^4},
\]

which is independent of \( \alpha_c, \alpha_s \). Therefore,

\[
f(\theta|\alpha_c, \alpha_s) = \frac{1}{\sqrt{2\pi\sigma^2_{\hat{\theta}|\alpha_c, \alpha_s}}} \exp\left( \frac{(\theta - \hat{\theta}_{\alpha_c, \alpha_s})^2}{2\sigma^2_{\hat{\theta}|\alpha_c, \alpha_s}} \right),
\]

which implies:

\[
P(\theta \leq \theta_{cr}|\alpha_c, \alpha_s) = \frac{\theta_{cr} - \hat{\theta}_{\alpha_c, \alpha_s}}{\sigma_{\hat{\theta}|\alpha_c, \alpha_s}} \int_{-\infty}^{\theta_{cr}} f(\theta|\alpha_c, \alpha_s) d\theta
\]

\[
= \Phi\left[ \frac{\theta_{cr} - \hat{\theta}_{\alpha_c, \alpha_s}}{\sigma_{\hat{\theta}|\alpha_c, \alpha_s}} \right]
\]

Therefore, the final expression for the conditional PDF of the wave amplitudes that do not cause critical deck submergence is:

\[
f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) = \Phi\left[ \frac{\theta_{cr} - \hat{\theta}_{\alpha_c, \alpha_s}}{\sigma_{\hat{\theta}|\alpha_c, \alpha_s}} \right] f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}).
\]

The resulted conditional probability \( P(\theta \leq \theta_{cr}|\alpha_c, \alpha_s) \), as well as the associated PDF \( f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) \) are presented in Fig. 8.

STATISTICAL STRUCTURE OF VBM WITH DECK EFFECTS

The final step of the analysis involves the computation of the PDF taking into account the deck effects. This is straightforward by computing analytically the derived distribution for the wave-induced VBM with the quadratic expansion (21) and the derived conditional PDF \( f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) \). Specifically, for the cumulative distribution function of the wave induced VBM under the condition of deck submergence up to the critical pitch angle is:

\[
F_{M_{W-VBM}}^{\text{deck}}(M) = \int_{D(M)} f(\alpha_c, \alpha_s|\theta \leq \theta_{cr}) d\alpha_c d\alpha_s,
\]

\[
D(M) = \{ (\alpha_c, \alpha_s): M_{W-VBM}(\alpha_c, \alpha_s) \leq M \}.
\]
effects. These are compared with the full VBM computed with LAMP. We observe how the conditioning to waves causing subcritical pitch angles results in a tail with finite heavy regime. The parameter \( q \) controls the location where the transition from heavy to light tail occurs. Here, a value of \( q = 0.44 \) results in the correct transition point.

**CONCLUSIONS**

The effect of the deck on the statistical distribution of the vertical bending moments was examined. While in the absence of deck, second order nonlinear effects lead to the formation of a heavy right tail, the existence of deck effects lead to the elimination of this behavior. To quantify this modification, an analytical model is developed that conditions the excitation waves based on whether they lead to deck submergence or not. First, the family of waves that lead to deck submergence is statistically quantified with a linear ship response model. Subsequently, the modified PDF for the vertical bending moment is derived analytically, by conditioning on the occurrence of waves that do not lead to significant deck submergence. The derived PDF is characterized by a minimal set of parameters, in particular just one: the critical pitch angle over which additional wave elevation does not cause a significant increase in the resulting PDF. Comparison with direct numerical simulations using LAMP confirm the accuracy of the derived model.

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**APPENDIX A: PROOF OF EQ. (16)**

Symmetry properties are applied to obtain

\[
\mathcal{J} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{s_0}{k - I\omega^2 + ic \omega} d\omega
\]

\[
= \frac{s_0}{\pi} \int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega
\]

\[
= \frac{s_0}{\pi} \int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega + \frac{s_0}{\pi} \int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega,
\]

where \( \omega_0 = \sqrt{\frac{k}{7}} \). The second integral on the right-hand side is transformed using \( \omega' = \frac{\omega_0}{\omega} \). In this way:

\[
\int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega = \int_{0}^{\infty} \frac{(k\omega^2 - I\omega_0^2) \omega_0^2}{(k\omega^2 - I\omega_0^2)^2 + c^2\omega_0^4\omega^2} d\omega'
\]

\[
= \int_{0}^{\infty} \frac{k (\omega^2 - \omega_0^2) \omega_0^2}{k(\omega^2 - \omega_0^2)^2 + c^2\omega_0^4\omega^2} d\omega'
\]

\[
= \int_{0}^{\infty} \frac{k^2(\omega^2 - \omega_0^2)^2 + c^2\omega_0^4\omega^2}{(k\omega^2 - k)^2 + c^2\omega^2} d\omega'
\]

\[
= \int_{0}^{\infty} \frac{(k\omega^2 - k)}{(k\omega^2 - k)^2 + c^2\omega^2} d\omega'
\]

Therefore,

\[
\mathcal{J} = \frac{s_0}{\pi} \int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega
\]

\[
- \frac{s_0}{\pi} \int_{0}^{\infty} \frac{k - I\omega^2}{(k - I\omega^2)^2 + (c \omega)^2} d\omega = 0.
\]

**REFERENCES**


