Computations in Riemannian space involving geodesics with their singularities and computing and visualizing singularities of dynamical systems using numerical methods based on differential geometry
The castle *Welfenschloss* (Leibniz University of Hanover)
The *HapTex* Project

- **Haptic Sensing of Virtual Textiles**
- "Virtual Reality System" consisting of
  - Tactile display
    - Sensation of fine surface structure
  - Haptical interaction
    - Larger scale force feedback
  - Visual representation
    - Simulation and visualization of virtual fabric

For background cf. 1, 2, 3 in List of references
Tactile Displays for Virtual Reality Applications

- **Aim**: Representation of arbitrary surfaces on the tactile display
- FE Analysis to establish receptor specific oscillation patterns
- Search for optimal control of the display
Research: Cochlea

Physiology of Inner Ear

Existing cochlear implants

3D Fluid Simulation

Evaluation and Optimization of hearing implants

For background cf. 4 in List of references
YADIV

Open Source Platform for

• Visualization
• Analysis and
• Interaction

with (medical) volume data

Virtual Reality support:
• Stereographic visualization
• Haptic interaction

For background cf. 5 in List of references
Laplace Spectra

Image clustering  Shape clustering  Eigenfunctions

For background cf. 6, 7, 8, 9 in List of references. For overall background cf. 10
Connecting Geodesics
Dynamics of Slow-Fast-Vector fields
Part I
Connecting Geodesics

For background cf. 10, 12, 13 in List of references. Please observe in particular reference 14.
Basic Setting

- Complete Riemannian Manifold \((M, g)\)
- Reference Point, Vector \(p \in M, v \in T_p M\)
- Geodesic \(\gamma_{p,v} : [0, \infty) \rightarrow M \quad \gamma_{p,v}(0) = p, \gamma_{p,v}'(0) = v\)
- Exponential map \(\exp_p : T_p M \rightarrow M \quad v \mapsto \gamma_{p,v}(1)\)
Distance Function

- Riemannian metric
  \[ p \mapsto (g_p : T_p M \times T_p M \rightarrow \mathbb{R}) \]

- Length functional
  \[ L : C([a, b], M) \rightarrow \mathbb{R} \quad \gamma \mapsto \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \]

- Geodesics are stationary points of \( L \)

- Distance function
  \[ d_M(p, q) = \inf \{ L(\gamma) | \gamma(a) = p, \gamma(b) = q \} \]
  \[ = \inf \{ L(\gamma) | \gamma \text{ geodesic }, \gamma(a) = p, \gamma(b) = q \} \]
Geodesic polar coordinates

- \((s, \varphi)\) polar coordinates in \(T_p M\)
- Exponential map
  \[\exp_p : T_p M \rightarrow M\]
  \[(s, \varphi) \mapsto \exp_p (s, \varphi)\]
- induces geodesic polar coordinates (GPC) on \(M\)
Parametric description

- Parameter representation of some curve $c : [0, 1] \rightarrow N \subset M$
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- Parameter representation of some curve $c : [0, 1] \rightarrow N \subset M$
- Known: GPCs of $c(0) = \exp_p(s_0, \varphi_0)$
- Wanted: GPCs of $c(t) = \exp_p(s(t), \varphi(t))$

$$\dot{c} = \left( \frac{\partial}{\partial s} \exp_p(s, \varphi) \quad \frac{\partial}{\partial \varphi} \exp_p(s, \varphi) \right) \begin{pmatrix} \dot{s} \\ \dot{\varphi} \end{pmatrix} \quad \text{(implicit ODE)}$$
Basic Idea of Geodesic Lift

- Curve $Q : [0,1] \rightarrow M$
- Geodesic to start point is known, i.e. $\theta_0 \in T_p M$ with $\exp_p(\theta_0) = Q(0)$
- Ansatz
  $$\exp_p(\theta(t)) = Q(t)$$
- Differentiating yields
  $$d\exp_p(\theta(t))\dot{\theta}(t) = \dot{Q}(t)$$
  $$\implies \dot{\theta}(t) = d\exp_p^{-1}(\theta(t))\dot{Q}(t)$$
  if $d\exp_p(\theta(t))$ is invertible
Generalization to Reference Submanifolds

- Closed reference submanifold \( A \subset M \)
- Normal bundle

\[
NA = \{ \theta \in TM | \pi(\theta) \in A, \theta \perp T_{\pi(\theta)}A \}
\]

- Restrict exponential map

\[
\exp_A : NA \rightarrow M
\]

- Singularities of \( \exp_p \) and \( \exp_A \) are structurally similar
Singularities of the Exponential Map

- Focal points of \( \exp_A \)

\[
F_A = \{ \theta \in NA | d \exp_A(\theta) \text{ singular} \}
\]

- Introduce unknown reparametrisation of \( Q \)

\( \lambda : [0, T] \rightarrow [0, 1] \)

- Extended ansatz: \( \exp_A(\theta(t)) - Q(\lambda(t)) = 0 \)

- Differentiating yields

\[
d \exp_A(\theta(t)) \dot{\theta}(t) - \dot{Q}(\lambda(t)) \dot{\lambda}(t) = 0
\]

\[
\implies \begin{pmatrix} \dot{\theta}(t) \\ \dot{\lambda}(t) \end{pmatrix} \in \ker[d \exp_A(\theta(t)) | - \dot{Q}(t)]
\]

For background cf. 10, 12, 13, 14
Classification of Focal Points

- **Nullspace**

\[ N_\theta = \{ w \in T_\theta N A | d \exp_A(\theta)[w] = 0 \} \]

- **\( F_k \)** = \( \{ \theta \in F | \dim N_\theta = k \} \)

\[ \implies F = F_1 \cup F_2 \cup F_3 \cup \ldots \]

- **\( F_1 \)** regular focal points, \( F_k, k > 1 \) irregular

- **\( F_1 \)** decomposes further into
  Folds, Cusps, Swallowtails, ...

- **\( F_2 \)** contains so-called Umbilics
Focal points in 2d and 3d Manifolds

- In 2d: Folds and Cusps
  - Folds: one-dimensional
  - Cusps: zero-dimensional, isolated

- In 3d: Folds, Cusps, Swallowtails and Umbilics
  - Folds: two-dimensional
  - Cusps: one-dimensional
  - Swallowtails, Umbilics: zero-dimensional, isolated

For background cf. 10, 12, 13, 14
3d Euclidean Example – Classification of Focal Points

- Reference surface in $\mathbb{R}^3$:
  \[ f(u,v) = (u,v,a \cos(u) + b \sin(v)) \]

  \[
  \begin{align*}
  a &= 0.5, b = 0.1 & a &= 0.5, b = 0.3 & a &= 0.5, b = 0.5
  \end{align*}
\]

  Folds, Cusps, Swallowtails + Umbilics

For background cf. 10, 12, 13, 14
3d Riemannian Example – Classification of Focal Points

- Reference surface in $M^3 \subset \mathbb{R}^4$:
  
  $$M^3 = \{(x, y, z, h(x, y, z))\}$$
  
  $$h(x, y, z) = \cos(x) + \sin(y) + \cos(z)$$

  $a = 0.5, b = 0.1$  
  $a = 0.5, b = 0.3$  
  $a = 0.5, b = 0.5$

Folds, Cusps, Swallowtails + Umbilics

For background cf. 10, 12, 13, 14
Interpretation of Geodesic Lift in Tangent Bundle

- Exponential map
  \[ \text{Exp}_A : NA \rightarrow TM \]
  \[ \theta \mapsto \dot{\gamma}_\theta(1) \]

- Geodesic flow manifold
  \[ G_A = \text{Exp}_A(NA) \]

- Trivial lift of \( Q \)
  \[ \pi^*Q = \{ \theta \in TM | \pi(\theta) \in Q \} \]

- Geodesic lift of \( Q \) w.r.t. \( A \)
  \[ G_A \cap \pi^*Q \]

For background cf. 10, 12, 13, 14
Applications – Medial

- Medial $m_{A,B}$ of two reference objects $A, B$
- Singularitites of $\exp_A, \exp_B$ affect $m_{A,B}$
- Initial point on $m_{A,B}$ via geodesic lift
- Classification of focal points applies
Applications – Voronoi Diagram

- Set of reference objects
  \[ O = A_1, \ldots, A_n \]
- Voronoi-Diagram is constructed from resp. medials

For background cf. 10, 12, 13, 14
Applications – Voronoi Diagram (2)

- Usage within incremental Voronoi algorithm
Part II

Dynamics of Slow-Fast-Vector fields

For background cf. 11 in List of references
Differential Algebraic Equation (DAE) System

Geometry: \[ 0 = f(x, y) \quad f : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n \]
Dynamic: \[ \dot{y} = g(x, y) \quad g : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n \]

- \( x = (x^1, \ldots, x^m) \) control variables
- \( y = (y^1, \ldots, y^n) \) jumping variables
- \( f \) defines submanifold
- \( g \) defines slow vector field \( X \)
Standard Cusp Catastrophe
Introductory example

\[ V_\varepsilon(x, y) = \left( \frac{1}{\varepsilon} f(x, y) \right) = \left( \frac{1}{\varepsilon} f(x, y) \right) + \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \]

\[ f(x, y) = -(x + (7 - y)^2 - 5)(y - 1) \]
Introductory example

\[ V_\varepsilon(x, y) = \begin{pmatrix} 1 \\ \frac{1}{\varepsilon} f(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\varepsilon} f(x, y) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ f(x, y) = -(x + (7 - y)^2 - 5)(y - 1) \]
Introductory example

\[ V_\varepsilon(x, y) = \left( \frac{1}{\varepsilon} f(x, y) \right) = \left( \frac{1}{\varepsilon} f(x, y) \right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ f(x, y) = -(x + (7 - y)^2 - 5)(y - 1) \]
Introductory example

\[ V_\varepsilon(x, y) = \frac{1}{\varepsilon} f(x, y) = \left( \frac{1}{\varepsilon} f(x, y) \right) + 1 \]

\[ f(x, y) = -(x + (7 - y)^2 - 5)(y - 1) \]
Concepts and notions

- **Attractor**
- **Repeller**
- **Jump set**
- **Hit set**
- **Jumps**
Jump set

- Projection
  \[ \pi : M \to \mathbb{R}^m \quad (x, y) \mapsto x \]

- Jump set
  \[ A = \left\{ (x, y) \in M \bigg| \text{Rank}(d\pi_{(x,y)}) < m \right\} \]

- Determinant criterion
  \( (x, y) \in A \iff \det \left( \frac{\partial f(x, y)}{\partial y} \right) = 0 \)

- Implicit representation
  \[ \begin{pmatrix} f(x, y) \\ \det \left( \frac{\partial f(x, y)}{\partial y} \right) \end{pmatrix} = 0 \]
Hit set

- **Jumping point**
  \[ q = (q_x, q_y) \in A \]

- **Jumping space**
  \[ Q_q = \{ (q_x, y) \in \mathbb{R}^{m+n} | y \in \mathbb{R}^n \} \]

- **Hit set**
  \[ B = \bigcup_{q \in A} (Q_q \cap M) \setminus A \]
(Numerical) geodesic polar coordinates (for implicitly defined manifolds)

- \((s, \varphi)\) polar coordinates in \(T_pM\)
- Exponential map  
  \[ \exp_p : T_pM \rightarrow M \quad (s, \varphi) \mapsto \exp_p(s, \varphi) \]
- induces geodesic polar coordinates (GPC) on \(M\)
Cut Locus

- Reference point \( p \in M \)
- Cut Locus \( C_p \subset M \) represents glueing seam in \( T_p M \)
Parametric description

- Parameter representation of some curve $c : [0, 1] \rightarrow N \subset M$
Parametric description

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- Known: GPCs of $c(0) = \exp_p(s_0, \varphi_0)$
- Wanted: GPCs of $c(t) = \exp_p(s(t), \varphi(t))$

$$\dot{c} = \left( \frac{\partial}{\partial s} \exp_p(s, \varphi) \frac{\partial}{\partial \varphi} \exp_p(s, \varphi) \right) \begin{pmatrix} \dot{s} \\ \dot{\varphi} \end{pmatrix} \quad \text{(implicit ODE)}$$
Reference point

- implicit representation $M = f^{-1}(0)$
- $p \in M \iff f(p) = 0$
- Find $p$ via root determination, e.g. homotopy method
Hit set

- Jumping point
  \[ q = (q_x, q_y) \in A \]

- Jumping space
  \[ Q_q = \{(q_x, y) \in \mathbb{R}^{m+n} \mid y \in \mathbb{R}^n\} \]

- Hit set
  \[ B = \bigcup_{q \in A} (Q_q \cap M) \setminus A \]
Correspondence: Jump- and hit set

\[ \alpha : \mathbb{R} \rightarrow A \quad \beta : \mathbb{R} \rightarrow B \]  
with \( \pi \circ \alpha = \pi \circ \beta \)

- \( \alpha \) e. g. geodesic in \( A \)

- Simultaneous calculation of GPCs yields
  \[ \alpha(t) = \exp_{\rho}(r(t), \psi(t)) \quad \beta(t) = \exp_{\rho}(s(t), \varphi(t)) \]
- Jump in GPC: \( (r(t), \psi(t)) \mapsto (s(t), \varphi(t)) \)
\[ K(r, \psi) := \det \left( \frac{\partial f(\gamma(r, \psi))}{\partial y} \right) = 0 \]

\[ F(s, \varphi, r, \psi) := \begin{pmatrix} K(r, \psi) \\ \pi(\gamma(r, \psi)) - \pi(\gamma(s, \varphi)) \end{pmatrix} = 0 \]

\[ H(q(t), \lambda(t)) = F(q(t)) + (\lambda(t) - 1)F(q_0) = 0 \]
**Example: Surface in** $\mathbb{R}^4$

\[
f_1(w) = w_3^3 - w_2 w_3^2 - \frac{1}{2} w_4 + \frac{\sqrt{3}}{2} w_1 = 0
\]

\[
f_2(w) = \frac{\sqrt{3}}{2} w_4 + \frac{1}{2} w_1 = 0
\]

\[
g_1(w, \dot{w}) = \dot{w}_1 + w_2 = 0
\]

\[
g_2(w, \dot{w}) = \dot{w}_2 - w_1 = 0
\]
Example: Surface in $\mathbb{R}^4$
**Example: Heartbeat**

\[
\varepsilon \dot{x} = -(x^3 + ax + b)
\]

\[
\dot{b} = x - x_{0,1}
\]

\[
f(a, b, x) = x^3 + ax + b
\]

\[
g(a, b, x, \dot{b}) = x - x_{0,1} - \dot{b}
\]

<table>
<thead>
<tr>
<th>Case</th>
<th>$a$</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$&gt; 0$</td>
<td>very low pressure</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>resting</td>
</tr>
<tr>
<td>3</td>
<td>$&lt; 0$</td>
<td>under load</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>elevated pressure</td>
</tr>
</tbody>
</table>
Example: Heartbeat

For background cf. 11. The original idea of this Heart Model appears to be due to C. Zeemann. Reference 11 reinterpretes C. Zeeman’s example within a new computational approach employing concepts from computational differential geometry as a global systematic framework used to analyse and visualize the respective dynamical systems.
Thank you for your attention
List of References

1. D. Allerkamp, G. Böttcher, F.-E. Wolter, A. C. Brady, J. Qu, I.A. Summers
   "A Vibrotactile Approach to Tactile Rendering"

2. D. Allerkamp
   "Tactile Perception of Textiles in a Virtual-Reality System"

3. G. Böttcher
   "Haptic Interaction with Deformable Objects"

4. C. Gerstenberger, F.-E. Wolter
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5. K.I. Friese, P. Blanke, F.-E. Wolter
   "YaDiV – an open platform for 3d visualization and 3d segmentation of medical data"
   The Visual Computer, vol. 27, p. 129-139 (2011)

6. F.-E. Wolter, N. Peinecke, M. Reuter
   "A Method for the Characterization of Objects (Surfaces, Solids and Images)"

7. N. Peinecke, F.-E. Wolter, M. Reuter
   "Laplace spectra as fingerprints for image recognition"

8. M. Reuter, F.-E. Wolter, N. Peinecke
   "Laplace-Beltrami Spectra as Shape DNA of Surfaces and Solids"

   "Laplacians on flat line bundles over 3-manifolds"

    "Computational Differential Geometry Contributions of the Welfenlab to GRK 615"

11. M. Gutschke, A. Vais, F.-E. Wolter
    "Differential Geometric Methods for examining the dynamics of slow–fast vector fields"

12. H. Thielhelm, A. Vais, F.-E. Wolter
    "Geodesic Bifurcation on Smooth Surfaces"

    "Connecting geodesics on smooth surfaces"

14. In part I of this seminar, (slide 10 - 26) all the computational results stated there incl. respective figures and animations are stemming from an ongoing Ph.D. research project of Hannes Thielhelm, Welfenlab, Leibniz Univ. of Hannover