Many phenomena in nature exhibit coherent structures

Unknown parameters, forcing or boundary conditions $\rightarrow$ Stochastic modeling $\rightarrow$ Need for efficient order reduction methods

*G* is a group of continuous transformations (translations and/or rotations)

Blending symmetry and dimensionality reduction

We introduce a new methodology for efficient order reduction of such systems:
1. **Symmetry reduction** of the original system, leading to a symmetry-reduced stochastic state and associated evolution equations
2. **Dimensionality reduction** of the symmetry-reduced state

**Step 1: Symmetry reduction**

For each realization, replace $\mathbf{u}$ by symmetry-reduced state $\hat{\mathbf{u}} = g^j(\Phi) \mathbf{u}$ such that distance $|| \hat{\mathbf{u}} - \mathbf{g}^j ||$ to a fixed template $\hat{\mathbf{u}}$ is minimized

$\hat{\mathbf{u}} = g^j(\Phi) \mathbf{u}$

Condition for $\hat{\mathbf{u}}$ and $\Phi$:

$\langle \mathbf{u}, g(\Phi) \mathbf{t}_i^j \rangle = (\hat{\mathbf{u}}, \mathbf{t}_i^j) = 0$

$\mathbf{t}_i^j = \lim_{\Delta \to 0} \frac{g(\Delta \mathbf{u})\mathbf{w} - \mathbf{w}}{\Delta \mathbf{u}}$

As $\mathbf{u}$ evolves under its dynamical governing equation, the evolution of $\hat{\mathbf{u}}$ and $\Phi$ is governed by

$\frac{\partial \hat{\mathbf{u}}}{\partial t} = \mathbf{F}(\hat{\mathbf{u}})$

$\frac{\partial \Phi}{\partial t} = \hat{\phi}_a(\hat{\mathbf{u}}, \mathbf{t}_i^j) = (\mathbf{F}(\hat{\mathbf{u}}), \mathbf{t}_i^j)$

**Step 2: Dimensionality reduction**

Stochastic dynamics of $\hat{\mathbf{u}}(\mathbf{x},t;\omega)$ lives in lower-dimensional space than $\mathbf{u}(\mathbf{x},t;\omega)$ $\rightarrow$ apply dimensionality reduction directly to $\hat{\mathbf{u}}(\mathbf{x},t;\omega)$

As an illustration, we use the Dynamically Orthogonal (DO) scheme

$\hat{\mathbf{u}}(\mathbf{x},t;\omega) = \mathbf{u}(\mathbf{x},t;\omega) + \sum_{g \in G} Y_{g}(t;\omega) \mathbf{u}_g(\mathbf{x};\omega)$

where $\mathbf{F}(\mathbf{u}) = gF(\mathbf{u}) \forall g \in G$

$G$ is a group of continuous transformations (translations and/or rotations)

$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, t; \omega)$

Insert into the governing equations for $\hat{\mathbf{u}}$ to obtain explicit equations for the mean, modes and coefficients $\rightarrow$ Symmetry-reduced (SDO) scheme

$\hat{\mathbf{u}}(\mathbf{x},t;\omega) = g(\Phi(\mathbf{x};t;\omega)) \hat{\mathbf{u}}(\mathbf{x},t;\omega)$

SDO is a nonlinear scheme!

Application to 2D Navier-Stokes equations

The NS equations in a periodic domain are invariant under translations

$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}$

Stochastic initial condition: Lamb-Oseen vortex of random size initially centered at the origin and undergoing random advection

$\mathbf{u}_0(\mathbf{x},y;\omega) = \frac{1}{2\pi} \left( 1 - e^{-x^2+y^2/\sigma(\omega)^2} \right) (-\sin \theta, \cos \theta)^T + (\cos \psi(\omega), \sin \psi(\omega))^T$

Initial condition

\[ \mathbf{u}(\mathbf{x},t;\omega) = \mathbf{u}_0(\mathbf{x},t;\omega) = \frac{1}{2\pi} \left( 1 - e^{-x^2+y^2/\sigma(\omega)^2} \right) (-\sin \theta, \cos \theta)^T + (\cos \psi(\omega), \sin \psi(\omega))^T \]

Reduced-order solution with Dynamically Orthogonal scheme - 4 modes

DO fails to preserve the shape of each realization

Current state-of-the-art

Example: Gaussian concentration profile advected by random velocity

$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V}(t;\omega) \frac{\partial \mathbf{u}}{\partial x} = 0$

SDO outperforms DO because its means and modes do not need to account for the stochastic translation of the realizations.