# Heavy-tailed response of structural systems subjected to stochastic excitation containing extreme forcing events

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**Abstract**

We characterize the complex, heavy-tailed probability distribution functions (pdf) describing the response and its local extrema for structural systems subjected to random forcing that includes extreme events. Our approach is based on the recent probabilistic decomposition-synthesis technique in [21], where we decouple rare events regimes from the background fluctuations. The result of the analysis has the form of a semi-analytical approximation formula for the pdf of the response (displacement, velocity, and acceleration) and the pdf of the local extrema. For special limiting cases (lightly damped or heavily damped systems) our analysis provides fully analytical approximations. We also demonstrate how the method can be applied to high dimensional structural systems through a two-degrees-of-freedom structural system undergoing rare events due to intermittent forcing. The derived formulas can be evaluated with very small computational cost and are shown to accurately capture the complicated heavy-tailed and asymmetrical features in the probability distribution many standard deviations away from the mean, through comparisons with expensive Monte-Carlo simulations.

**Keywords** Rare and extreme events; Intermittently forced structural systems; Heavy-tails; Colored stochastic excitation; Random impulse trains.

## 1 Introduction

A large class of physical systems in engineering and science can be modeled by stochastic differential equations. For many of these systems, the dominant source of uncertainty is due to the forcing which can be described by a stochastic process. Applications include ocean engineering systems excited by water waves (such as ship motions in large waves [23, 5, 9, 8] or high speed crafts subjected to rough seas [26, 25]) and rare events in structural systems (such as beam buckling [1, 17], vibrations due to earthquakes [16, 7] and wind loads [18, 31]). For all of these cases it is common that hidden in the otherwise predictable magnitude of the fluctuations are extreme events, i.e. abnormally large magnitude forces which lead to rare responses in the dynamics of the system (figure 1). Clearly, these events must be adequately taken into account

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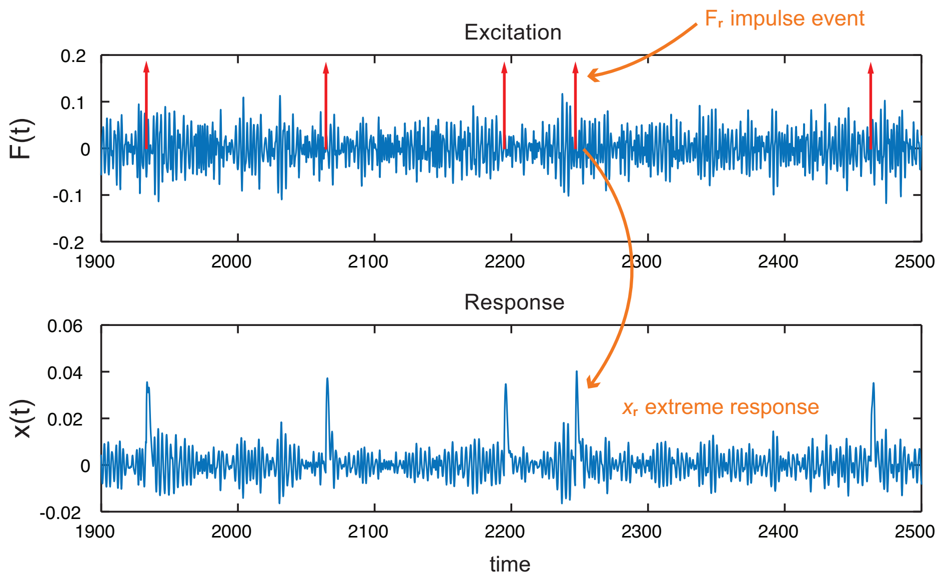


Figure 1: (Top) Background stochastic excitation including impulsive loads in (red) upward arrows. (Bottom) System response displacement.

for the effective quantification of the reliability properties of the system. In this work, we develop an efficient method to fully describe the probabilistic response of linear structural systems under general time-correlated random excitations containing rare and extreme events.

Systems with forcing having these characteristics pose significant challenges for traditional uncertainty quantification schemes. While there is a large class of methods that can accurately resolve the statistics associated with random excitations (e.g. the Fokker-Planck equation [30, 29] for systems excited by white-noise and the joint response-excitation method [28, 34, 12, 3] for arbitrary stochastic excitation) these have important limitations for high dimensional systems. In addition, even for low-dimensional systems determining the part of the probability density function (pdf) associated with extreme events poses important numerical challenges. On the other hand, Gaussian closure schemes and moment equation or cumulant closure methods [6, 35] either cannot "see" the rare events completely or they are very expensive and require the solution of an inverse moment problem in order to determine the pdf of interest [2]. Similarly, approaches relying on polynomial-chaos expansions [37, 36] have been shown to have important limitations for systems with intermittent responses [19].

Another popular approach for the study of rare event statistics in systems under intermittent forcing is to represent extreme events in the forcing as identically distributed independent impulses arriving at random times. The generalized Fokker-Planck equation or Kolmogorov-Feller (KF) equation is the governing equation that solves for the evolution of the response pdf under Poisson noise [29]. However, exact analytical solutions are available only for a limited number of special cases [33]. Although alternative methods such as the path integral method [14, 11, 4] and the stochastic averaging method [39, 38] may be applied, solving the FP or KF equations is often very expensive [20, 10] even for very low dimensional systems.

Here, we consider the problem of quantification of the response pdf and the pdf associated with local extrema of linear systems subjected to stochastic forcing containing extreme events based on the recently formulated probabilistic-decomposition synthesis (PDS) method [21, 22]. The approach relies on the decomposition of the statistics into a 'non-extreme core', typically

Gaussian, and a heavy-tailed component. This decomposition is in full correspondence with a partition of the phase space into a 'stable' region where we do not have rare events and a region where non-linear instabilities or external forcing lead to rare transitions with high probability. We quantify the statistics in the stable region using a Gaussian approximation approach, while the non-Gaussian distribution associated with the intermittently unstable regions of phase space is performed taking into account the non-trivial character of the dynamics (either because of instabilities or external forcing). The probabilistic information in the two domains is analytically synthesized through a total probability argument.

We begin with the simplest case of a linear, single-degree-of-freedom (SDOF) system and then formulate the method for multi-degree-of-freedom systems. *The main result of our work is the derivation of analytic/semi-analytic approximation formulas for the response pdf and the pdf of the local extrema of intermittently forced systems that can accurately characterize the statistics many standard deviations away from the mean.* Although the systems considered in this work are linear the method is directly applicable for nonlinear structural systems as well. This approach circumvents the challenges that rare events pose for traditional uncertainty quantification schemes, in particular the computational burden associated when dealing with rare events in systems. We emphasize the statistical accuracy and the computational efficiency of the presented approach, which we rigorously demonstrate through extensive comparisons with direct Monte-Carlo simulations. In brief, the principal contributions of this paper are:

- Analytical (under certain conditions) and semi-analytical (under no restrictions) pdf expressions for the response displacement, velocity and acceleration for single-degree-of-freedom systems under intermittent forcing.

- Semi-analytical pdf expressions for the value and the local extrema of the displacement, velocity and acceleration for multi-degree-of-freedom systems under intermittent forcing.

The paper is structured as follows. In section 2 we provide a general formulation of the probabilistic decomposition-synthesis method for the case of structural systems under intermittent forcing. Next, in section 3, we apply the developed method analytically, which is possible for two limiting cases: underdamped systems with $\zeta \ll 1$ or overdamped with $\zeta \gg 1$, where $\zeta$ is the damping ratio. The system we consider is excited by a forcing term consisting of a background time-correlated stochastic process superimposed with a random impulse train (describing the rare and extreme component). We give a detailed derivation of the response pdf of the system (displacement, velocity and acceleration) and compare the results with expensive Monte-Carlo simulations. In section 4, we slightly modify the developed formulation to derive a semi-analytical scheme considering the same linear system but without any restriction on the damping ratio $\zeta$, demonstrating global applicability of our approach. In section 5, we demonstrate applicability of our method for multiple-degree-of-freedom systems and in section 6 we present results for the local extremes of the response. Finally, we offer concluding remarks in section 7.

# 2 The probabilistic decomposition-synthesis method for intermittently forced structural systems

We provide with a brief presentation of the recently developed probabilistic decomposition-synthesis (PDS) method adapted for the case of intermittently forced linear structural systems [21]. We consider the following vibrational system,

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t), \quad \mathbf{x}(t) \in \mathbb{R}^n, \tag{1}$$

where $M$ is a mass matrix, $D$ is the damping matrix, and $K$ is the stiffness matrix. We assume $\mathbf{F}(t)$ is a stochastic forcing with intermittent characteristics that can be expressed as

$$\mathbf{F}(t) = \mathbf{F}_b(t) + \mathbf{F}_r(t). \tag{2}$$

The forcing consists of a background component $\mathbf{F}_b$ of characteristic magnitude $\sigma_b$ and a *rare and extreme* component $\mathbf{F}_r$ with magnitude $\sigma_r \gg \sigma_b$. The components $\mathbf{F}_b$ and $\mathbf{F}_r$ may both be (weakly) stationary stochastic processes, while the sum of the two processes can be, in general, non-stationary. This can be seen if we directly consider the sum of two (weakly) stationary processes $x_1$ and $x_2$, with time correlation functions $\mathrm{Corr}_{x_1}(\tau)$ and $\mathrm{Corr}_{x_2}(\tau)$, respectively. Then for the sum $z = x_1 + x_2$ we have

$$\mathrm{Corr}_z(t,\tau) = \mathrm{Corr}_{x_1}(\tau) + \mathbb{E}[x_1(t)x_2(t+\tau)] + \mathbb{E}[x_1(t+\tau)x_2(t)] + \mathrm{Corr}_{x_2}(\tau).$$

Therefore, the process $z$ is stationary if and only if the cross-covariance terms $\mathbb{E}[x_1(t)x_2(t+\tau)]$ and $\mathbb{E}[x_1(t+\tau)x_2(t)]$ are functions of $\tau$ only or they are zero (i.e. $x_1$ and $x_2$ are not correlated).

For the case where the excitation is given in terms of realizations, i.e. time-series, one can first separate the extreme events from the stationary background by applying time-frequency analysis methods (e.g. wavelets, [32]). Then the stationary background can be approximated with a Gaussian stationary stochastic process (with properly tuned covariance function) while the rare event component can be represented with a Poisson process with properly chosen parameters that represent the characteristics of the extreme forcing events (frequency and magnitude).

To apply the PDS method we also decompose the response into two terms

$$\mathbf{x}(t) = \mathbf{x}_b(t) + \mathbf{x}_r(t), \tag{3}$$

where $\mathbf{x}_b$ accounts for the background state (non-extreme) and $\mathbf{x}_r$ captures extreme responses (due to the intermittent forcing) – see figure 2. More precisely $\mathbf{x}_r$ is the system response under two conditions: (1) the forcing is given by $\mathbf{F} = \mathbf{F}_r$ (i.e. we have an impulse) and (2) the norm of the response is greater than the background response fluctuations according to a given criterion, e.g. $\|\mathbf{x}\| > \gamma$. However, as we will see in the following sections other criteria may be used. These rare transitions occur when we have an impulse and they also include a phase that relaxes the system back to the background state $\mathbf{x}_b$. The background component $\mathbf{x}_b$ corresponds to system response without rare events $\mathbf{x}_b = \mathbf{x} - \mathbf{x}_r$, and in this regime the system is primarily governed by the background forcing term $\mathbf{F}_b$.

We require that rare events are statistically independent from each other. In the generic formulation of the PDS we also need to assume that rare events have negligible effects on the background state $\mathbf{x}_b$ but here this assumption is not necessary due to the linear character of the examples considered. However, in order to apply the method for general nonlinear structural systems we need to have this condition satisfied. We also need the dynamics to be ergodic while the statistics we aim to approximate refer to long time averages [21].

Next, we focus on the statistical characteristics of an individual mode $u(t) \in \mathbb{R}$ of the original system in equation (1). The first step of the PDS method is to quantify the conditional statistics of the rare event regime. When the system enters the rare event response at $t = t_0$ we will have an arbitrary background state $u_b$ as an initial condition at $t_0$ and the problem will be formulated as:

$$\ddot{u}_r(t) + \lambda\dot{u}_r(t) + ku_r(t) = F_r(t), \qquad \text{with } u_r(t_0) = u_b \text{ and } F = F_r \text{ for } t > t_0. \tag{4}$$

Under the assumption of independent rare events we can use equation (4) as a basis to derive analytical or numerical estimates for the statistical response during the rare event regime.

The background component, on the other hand, can be studied through the equation,

$$M\ddot{\mathbf{x}}_b(t) + D\dot{\mathbf{x}}_b(t) + K\mathbf{x}_b(t) = \mathbf{F}_b(t). \tag{5}$$

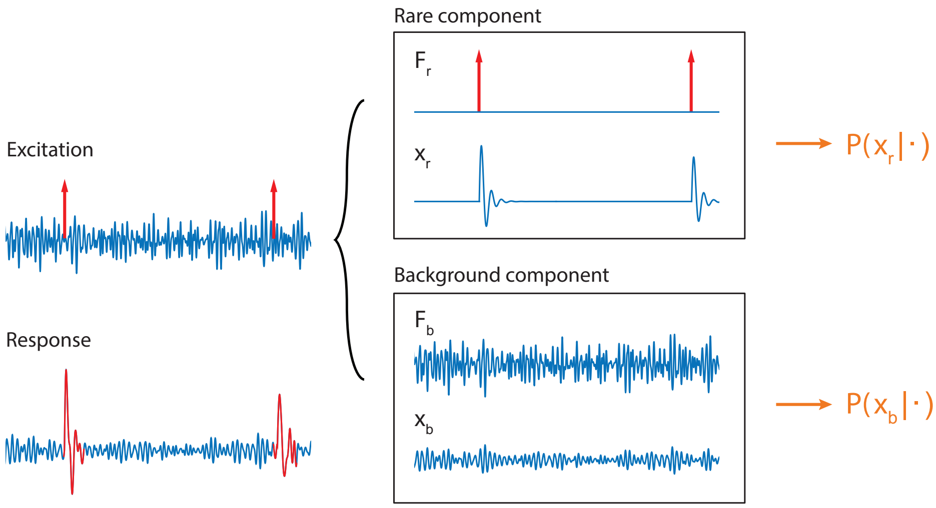


Figure 2: Schematic representation of the PDS method for an intermittently forced system.

Because of the non-intermittent character of the response in this regime, it is sufficient to obtain the low-order statistics of this system. In the context of vibrations it is reasonable to assume that $F_b(t)$ follows a Gaussian distribution in which case the problem is straightforward. Consequently, this step provides us with the statistical steady state probability distribution for the mode of interest under the condition that the dynamics 'live' in the stochastic background.

Finally, after the analysis of the two regimes is completed we can synthesize the results through a total probability argument

$$f(q) = \underbrace{f(q \mid \|u\| > \gamma, F = F_r)}_{\text{rare events}} \mathbb{P}_r + \underbrace{f(q \mid F = F_b)}_{\text{background}} (1 - \mathbb{P}_r), \tag{6}$$

where $q$ may be any function of interest involving the response. In the last equation, $\mathbb{P}_r$ denotes the overall rare event probability. This is defined as the probability of the response exceeding a threshold $\gamma$ because of a rare event in the excitation:

$$\mathbb{P}_r \equiv \mathbb{P}(\|u\| > \gamma, F = F_r) = \frac{1}{T} \int\limits_{t \in T} \mathbb{1}(\|u\| > \gamma, F = F_r)\, dt, \tag{7}$$

where $\mathbb{1}(\,\cdot\,)$ is the indicator function. The rare event probability measures the total duration of the rare events taking into account their frequency and duration. The utility of the presented decomposition is its flexibility in capturing rare responses, since we can account for the rare event dynamics directly and connect their statistical properties directly to the original system response.

## 2.1 Problem formulation for linear SDOF systems

To demonstrate the method we begin with a very simple example and we consider a single-degree-of-freedom linear system

$$\ddot{x} + \lambda \dot{x} + kx = F(t), \tag{8}$$

where $k$ is the stiffness, $\lambda$ is the damping, $\zeta = \lambda/2\sqrt{k}$ is the damping ratio. For what follows we adopt the standard definitions: $\omega_n = \sqrt{k}$, $\omega_o = \omega_n\sqrt{\zeta^2 - 1}$, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. $F(t)$ is a stochastic forcing term with intermittent characteristics, which can be written as

$$F(t) = F_b(t) + F_r(t). \tag{9}$$

Here $F_b$ is the background forcing component that has a characteristic magnitude $\sigma_b$ and $F_r$ is a rare and large amplitude forcing component that has a characteristic magnitude $\sigma_r$, which is much larger than the magnitude of the background forcing, $\sigma_r \gg \sigma_b$. Despite the simplicity of the system, its response may feature a significantly complicated statistical structure with heavy-tailed characteristics.

For concreteness, we consider a prototype system motivated from ocean engineering applications, modeling base excitation of a structural mode:

$$\ddot{x} + \lambda \dot{x} + kx = \ddot{h}(t) + \sum_{i=1}^{N(t)} \alpha_i \, \delta(t - \tau_i), \quad 0 < t \leq T. \tag{10}$$

Here $h(t)$ denotes the zero-mean background base motion term (having opposite sign from $x$) with a Pierson-Moskowitz spectrum:

$$S_{hh}(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right), \tag{11}$$

where $q$ controls the magnitude of the forcing.

The second forcing term in equation (10) describes rare and extreme events. In particular, we assume this component is a random impulse train ($\delta(\,\cdot\,)$ is a unit impulse), where $N(t)$ is a Poisson counting process that represents the number of impulses that arrive in the time interval $0 < t \leq T$, $\alpha$ is the impulse mean magnitude (characterizing the rare event magnitude $\sigma_r$), which we assume is normally distributed with mean $\mu_\alpha$, variance $\sigma_\alpha^2$ and independent from the state of the system. In addition, the arrival rate is constant and given by $\nu_\alpha$ (or by the mean arrival time $T_\alpha = 1/\nu_\alpha$ so that impulse arrival times are exponentially distributed $\tau \sim e^{T_\alpha}$).

We take the impulse mean magnitude as being $m$-times larger than the standard deviation of the excitation velocity $\dot{h}(t)$:

$$\mu_\alpha = m\sigma_{\dot{h}}, \text{ with } m > 1, \tag{12}$$

where $\sigma_{\dot{h}}$ is the standard deviation of $\dot{h}(t)$. This prototype system is widely applicable to numerous applications, including structures under wind excitations, systems under seismic excitations, and vibrations of high-speed crafts and road vehicles [29, 30, 26].

# 3 Analytical pdf of SDOF systems for limiting cases

In this section, we apply the probabilistic decomposition-synthesis method for the special cases $\zeta \ll 1$ and $\zeta \gg 1$ to derive analytical approximations for the pdf of the displacement, velocity and acceleration. We perform the analysis first for the response displacement and by way of a minor modification obtain the pdf for the velocity and acceleration.

## 3.1 Background response pdf

Consider the statistical response of the system to the background forcing component,

$$\ddot{x}_b + \lambda \dot{x}_b + kx_b = \ddot{h}(t), \tag{13}$$

due to the Gaussian character of the statistics the response is fully characterized by it's spectral response. The spectral density of the displacement, velocity and acceleration of this system are given by,

$$S_{x_b x_b}(\omega) = \frac{\omega^4 S_{hh}(\omega)}{\{k - \omega^2 + \lambda(j\omega)\}^2}, \quad S_{\dot{x}_b \dot{x}_b}(\omega) = \omega^2 S_{x_b x_b}(\omega), \quad S_{\ddot{x}_b \ddot{x}_b}(\omega) = \omega^4 S_{x_b x_b}(\omega). \tag{14}$$

Thus, we can obtain the variance of the response displacement, velocity and acceleration:

$$\sigma^2_{x_b} = \int_0^\infty S_{x_b x_b}(\omega)\, d\omega, \quad \sigma^2_{\dot{x}_b} = \int_0^\infty S_{\dot{x}_b \dot{x}_b}(\omega)\, d\omega, \quad \sigma^2_{\ddot{x}_b} = \int_0^\infty S_{\ddot{x}_b \ddot{x}_b}(\omega)\, d\omega. \tag{15}$$

Moreover, the envelopes are Rayleigh distributed [15]:

$$u_b \sim \mathcal{R}(\sigma_{x_b}), \quad \dot{u}_b \sim \mathcal{R}(\sigma_{\dot{x}_b}), \quad \ddot{u}_b \sim \mathcal{R}(\sigma_{\ddot{x}_b}). \tag{16}$$

## 3.2 Analytical pdf for the underdamped case $\zeta \ll 1$

Because of the underdamped character of the response for the case of $\zeta \ll 1$, we focus on deriving the statistics of the local extrema. To this end, we will be presenting results for the statistics of the envelope of the response.

### 3.2.1 Rare events response

To estimate the rare event response we take into account the non-zero background velocity of the system $\dot{x}_b$ at the moment of impact, as well as the magnitude of the impact, $\alpha$. The actual value of the response $x_b$ is considered negligible. For this case, taking into account $\zeta \ll 1$, we have the envelopes of the response (displacement, velocity, acceleration) during the rare event given by (see appendix A for details),

$$u_r(t) \simeq \frac{|\dot{x}_b + \alpha|}{\omega_d} \mathrm{e}^{-\zeta\omega_n t}, \quad \dot{u}_r(t) \simeq |\dot{x}_b + \alpha| \mathrm{e}^{-\zeta\omega_n t}, \quad \ddot{u}_r(t) \simeq \omega_d |\dot{x}_b + \alpha| \mathrm{e}^{-\zeta\omega_n t}. \tag{17}$$

In equation (17) the two contributions $\dot{x}_b$ and $\alpha$ in the term $\dot{x}_b + \alpha$ are both Gaussian distributed and independent and therefore their sum is also Gaussian distributed as:

$$\eta \equiv \dot{x}_b + \alpha \sim \mathcal{N}(\mu_\alpha,\, \sigma^2_{\dot{x}_b} + \sigma^2_\alpha). \tag{18}$$

Therefore, the distribution of the quantity $|\eta|$ is given by the following folded normal distribution:

$$f_{|\eta|}(n) = \frac{1}{\sigma_{|\eta|}\sqrt{2\pi}} \left\{ \exp\left( -\frac{(n-\mu_\alpha)^2}{2\sigma^2_{|\eta|}} \right) + \exp\left( -\frac{(n+\mu_\alpha)^2}{2\sigma^2_{|\eta|}} \right) \right\}, \quad 0 < n < \infty, \tag{19}$$

where $\sigma_{|\eta|} = \sqrt{\sigma^2_{\dot{x}_b} + \sigma^2_\alpha}$.

### 3.2.2 Rare event probability

Next, we compute the rare event probability, which is the total duration of the rare events over a time interval, defined in equation (7). This will be done by employing an appropriate description for extreme events. One possible option is to set an absolute threshold $\gamma$. However, in the current context it is more convenient to set this threshold relative to the local maximum of the response. Specifically, the time duration $\tau_e$ a rare response takes to return back to the background state will be given by the duration starting from the initial impulse event time ($t_0$) to the point where the response has decayed back to $\rho_c$ (or $100\rho_c\%$) of its absolute maximum;

here and throughout this manuscript we take $\rho_c = 0.1$. This is a value that we considered without any tuning. We emphasize that the derived approximation is not sensitive to the exact value of $\rho_c$ as long as this has been chosen within reasonable values.

This means that $\tau_e$ is defined by

$$u_r(\tau_e + t_0) = \rho_c u_r(t_0), \tag{20}$$

We solve the above *using the derived envelopes* to obtain

$$\tau_e = -\frac{1}{\zeta\omega_n} \log \rho_c. \tag{21}$$

We note that due to the linear character of the system *the typical duration $\tau_e$ is independent of the background state or the impact intensity.* With the obtained value for $\tau_e$ we compute the probability of rare events using the frequency $\nu_\alpha$ (equal to $1/T_\alpha$) :

$$\mathbb{P}_r = \nu_\alpha \tau_e = \tau_e / T_\alpha. \tag{22}$$

Note that based on our assumption that extreme events are rare enough to be statistical independent, the above probability is much smaller than one.

### 3.2.3 Conditional pdf for rare events

We now proceed with the derivation of the pdf in the rare event regime. Consider again the response displacement during a rare event,

$$u_r(t) \sim \frac{|\eta|}{\omega_d} \mathrm{e}^{-\zeta\omega_n t'}, \tag{23}$$

here $t'$ is a random variable uniformly distributed between the initial impulse event and the end time $\tau_e$ (equation (21)) when the response has relaxed back to the background dynamics:

$$t' \sim \text{Uniform}(0,\, \tau_e). \tag{24}$$

We condition the rare event distribution as follows,

$$f_{u_r}(r) = \int f_{u_r||\eta|}(r \mid n) f_{|\eta|}(n)\, dn, \tag{25}$$

where we have already derived the pdf for $f_{|\eta|}$ in equation (19). What remains is the derivation of the conditional pdf for $f_{u_r||\eta|}$.

By conditioning on $|\eta| = n$, we find the derived distribution for the conditional pdf given by

$$f_{u_r||\eta|}(r \mid n) = \frac{1}{r\zeta\omega_n\tau_e} \left\{ s\left( r - \frac{n}{\omega_d} \mathrm{e}^{-\zeta\omega_n\tau_e} \right) - s\left( r - \frac{n}{\omega_d} \right) \right\}, \tag{26}$$

where $s(\,\cdot\,)$ denotes the step function, which is equal to 1 when the argument is greater or equal to 0 and 0 otherwise. We refer to appendix B for a detailed derivation.

Using the equations (18) and (26), in equation (25) we obtain the final result for the rare event distribution for response displacement as

$$f_{u_r}(r) = \int f_{u_r||\eta|}(r \mid n) f_{|\eta|}(n)\, dn, \tag{27}$$

$$= \frac{1}{r\zeta\omega_n\sigma_{|\eta|}\tau_e\sqrt{2\pi}} \int_0^\infty \left\{ \exp\left( -\frac{(n-\mu_\alpha)^2}{2\sigma_{|\eta|}^2} \right) + \exp\left( -\frac{(n+\mu_\alpha)^2}{2\sigma_{|\eta|}^2} \right) \right\} \times \left\{ s\left( r - \frac{n}{\omega_d} \mathrm{e}^{-\zeta\omega_n\tau_e} \right) - s\left( r - \frac{n}{\omega_d} \right) \right\} dn. \tag{28}$$

#### 3.2.4 Summary of results for the underdamped case

**Displacement Envelope** Finally, combining the results of sections 3.1, 3.2.2, and 3.2.3 using the total probability law,

$$f_u(r) = f_{u_b}(r)(1-\mathbb{P}_r) + f_{u_r}(r)\mathbb{P}_r, \tag{29}$$

we obtain the desired envelope distribution for the displacement of the response

$$\begin{aligned} f_u(r) = {} & \frac{r}{\sigma_{x_b}^2}\exp\left(-\frac{r^2}{2\sigma_{x_b}^2}\right)(1-\nu_\alpha\tau_e) \\ & + \frac{\nu_\alpha\tau_e}{r\zeta\omega_n\sigma_{|\eta|}\tau_e\sqrt{2\pi}}\int_0^\infty\left\{\exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)+\exp\left(-\frac{(n+\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)\right\} \\ & \times\left\{s\left(r-\frac{n}{\omega_d}\mathrm{e}^{-\zeta\omega_n\tau_e}\right)-s\left(r-\frac{n}{\omega_d}\right)\right\}dn, \end{aligned} \tag{30}$$

where $\tau_e = -\frac{1}{\zeta\omega_n}\log\rho_c$ and $s(\,\cdot\,)$ denotes the step function.

**Velocity Envelope** Similarly, we obtain the envelope distribution of the velocity of the system. The background dynamics distribution for velocity was obtained in equation (16). Noting that from (17), $\dot{u}_r = \omega_d u_r$, the rare event pdf is modified by a constant factor

$$f_{\dot{u}}(r) = f_{\dot{u}_b}(r)(1-\mathbb{P}_r) + \omega_d^{-1} f_{u_r}(r/\omega_d)\mathbb{P}_r. \tag{31}$$

The final formula for the velocity envelope pdf is given by

$$\begin{aligned} f_{\dot{u}}(r) = {} & \frac{r}{\sigma_{\dot{x}_b}^2}\exp\left(-\frac{r^2}{2\sigma_{\dot{x}_b}^2}\right)(1-\nu_\alpha\tau_e) \\ & + \frac{\nu_\alpha}{r\zeta\omega_n\sigma_{|\eta|}\sqrt{2\pi}}\int_0^\infty\left\{\exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)+\exp\left(-\frac{(n+\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)\right\} \\ & \times\left\{s\left(r-n\mathrm{e}^{-\zeta\omega_n\tau_e}\right)-s\left(r-n\right)\right\}dn. \end{aligned} \tag{32}$$

**Acceleration Envelope** Lastly we also obtain the envelope distribution of the acceleration. Noting that from eq. (17), $\ddot{u}_r = \omega_d^2 u_r$, the rare event pdf for acceleration is also modified by a constant factor

$$f_{\ddot{u}}(r) = f_{\ddot{u}_b}(r)(1-\mathbb{P}_r) + \omega_d^{-2} f_{u_r}(r/\omega_d^2)\mathbb{P}_r. \tag{33}$$

The final formula for the acceleration envelope pdf is then

$$\begin{aligned} f_{\ddot{u}}(r) = {} & \frac{r}{\sigma_{\ddot{x}_b}^2}\exp\left(-\frac{r^2}{2\sigma_{\ddot{x}_b}^2}\right)(1-\nu_\alpha\tau_e) \\ & + \frac{\nu_\alpha}{r\zeta\omega_n\sigma_{|\eta|}\sqrt{2\pi}}\int_0^\infty\left\{\exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)+\exp\left(-\frac{(n+\mu_\alpha)^2}{2\sigma_{|\eta|}^2}\right)\right\} \\ & \times\left\{s\left(r-n\omega_d\mathrm{e}^{-\zeta\omega_n\tau_e}\right)-s\left(r-n\omega_d\right)\right\}dn. \end{aligned} \tag{34}$$

#### 3.2.5 Comparison with Monte-Carlo simulations

For the Monte-Carlo simulations the excitation time series is generated by superimposing the background and rare event components. The background excitation, described by a stationary stochastic process with a Pierson-Moskowitz spectrum (equation (11)), is simulated through a superposition of cosines over a range of frequencies with corresponding amplitudes and uniformly distributed random phases. The intermittent component is the random impulse train, and each impact is introduced as a velocity jump with a given magnitude at the point of the impulse impact. For each of the comparisons performed in this work we generate 10 realizations of the excitation time series, each with a train of 100 impulses. Once each ensemble time series for the excitation is computed, the governing ordinary differential equations are solved using a 4th/5th order Runge-Kutta method (we carefully account for the modifications in the momentum that an impulse imparts by integrating up to each impulse time and modifying the initial conditions that the impulse imparts before integrating the system to the next impulse time). For each realization the system is integrated for a sufficiently long time so that we have converged response statistics for the displacement, velocity, and acceleration.

We utilize a shifted Pierson-Moskowitz spectrum $S_{hh}(\omega - 1)$ in order to avoid resonance. The other parameters and resulted statistical quantities of the system are given in table 1. As it can be seen in figure 3 the analytical approximations compare well with the Monte-Carlo simulations many standard deviations away from the zero mean. The results are robust to different parameters as far as we satisfy the assumption of independent (non- overlapping) random events.

Some discrepancies shown between Monte-Carlo simulations can be attributed to the envelope approximation used for the rare event quantification. Indeed, these discrepancies are reduced significantly if one utilizes the semi-analytical method presented in the next section, where we do not make any simplifications for the form of the response during extreme events.

Table 1: Parameters and relevant statistical quantities for SDOF system 1.

| | | | |
|---|---|---|---|
| $\lambda$ | 0.01 | $k$ | 1 |
| $T_\alpha$ | 5000 | $\zeta$ | 0.005 |
| $\omega_n$ | 1 | $\omega_d$ | 1 |
| $\mu_\alpha = 7 \times \sigma_{\dot{h}}$ | 0.1 | $q$ | $1.582 \times 10^{-4}$ |
| $\sigma_\alpha = \sigma_{\dot{h}}$ | 0.0143 | $\sigma_h$ | 0.0063 |
| $\sigma_{\dot{x}_b}$ | 0.0179 | $\sigma_{x_b}$ | 0.0082 |
| $\sigma_{\lvert\eta\rvert}$ | 0.0229 | $\mathbb{P}_r$ | 0.0647 |

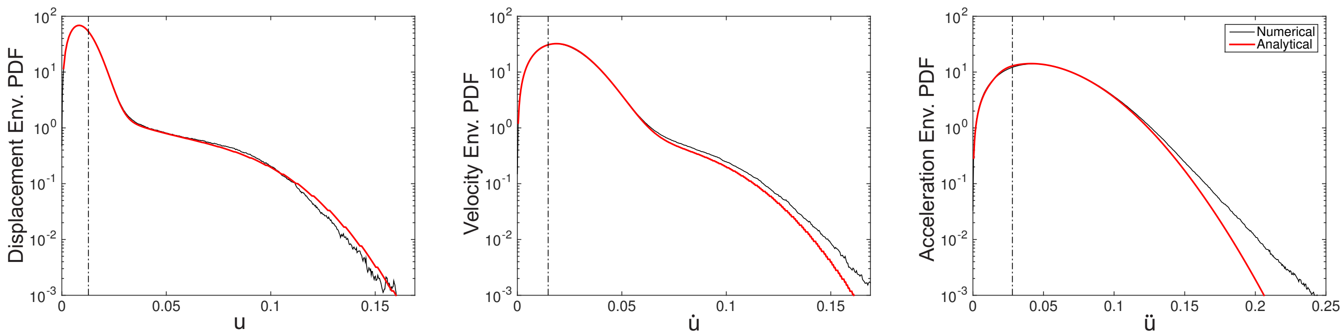


Figure 3: [**Severely underdamped case**] Comparison between direct Monte-Carlo simulation and the analytical pdf for the SDOF system 1. The pdf for the envelope of each of the stochastic variables, displacement, velocity, and acceleration, are presented. The dashed line indicates one standard deviation.

### 3.3 Analytical pdf for the overdamped case $\zeta \gg 1$

In the previous section, we illustrated the derivation of the analytical response pdf under the assumption $\zeta \ll 1$. Here, we briefly summarize the results for the response pdf for the case where $\zeta \gg 1$. One can follow the same steps using the corresponding formulas for the rare event transitions in the presence of large damping (appendix A). *An important difference for the overdamped case is that the system does not exhibit highly oscillatory motion as opposed to the underdamped case, and hence we directly work on the response pdf instead of the envelope pdf.*

**Displacement** The total probability law becomes

$$f_x(r) = f_{x_b}(r)(1 - \mathbb{P}_{r,\text{dis}}) + f_{x_r}(r)\mathbb{P}_{r,\text{dis}}, \tag{35}$$

and we obtain the following pdf for the displacement of the system

$$\begin{aligned} f_x(r) = &\frac{1}{\sigma_{x_b}\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma_{x_b}^2}\right)(1 - \nu_\alpha \tau_{e,\text{dis}}) \\ &+ \frac{\nu_\alpha \tau_{e,\text{dis}}}{r(\zeta\omega_n - \omega_o)\sigma_\eta \sqrt{2\pi}\,(\tau_{e,\text{dis}} - \tau_s)} \int_{-\infty}^{\infty} \exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_\eta^2}\right) \\ &\times \left\{ s\left(r - \frac{n}{2\omega_o} \mathrm{e}^{-(\zeta\omega_n - \omega_o)\tau_{e,\text{dis}}}\right) - s\left(r - \frac{n}{2\omega_o}\right)\right\} dn, \end{aligned} \tag{36}$$

where $\tau_{e,\text{dis}} = \frac{\pi}{2\omega_o} - \frac{1}{\zeta\omega_n - \omega_o}\log\rho_c$.

**Velocity** Similarly we derive the total probability law for the response velocity

$$f_{\dot{x}}(r) = f_{\dot{x}_b}(r)(1 - \mathbb{P}_{r,\text{vel}}) + f_{\dot{x}_r}(r)\mathbb{P}_{r,\text{vel}}. \tag{37}$$

The final result for the velocity pdf is

$$
\begin{aligned}
f_{\dot{x}}(r) = \frac{1}{\sigma_{\dot{x}_b}\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma_{\dot{x}_b}^2}\right)(1-\nu_\alpha \tau_{e,\text{vel}}) \\
+ \frac{\nu_\alpha \tau_{e,\text{vel}}}{r(\zeta\omega_n+\omega_o)\sigma_\eta\sqrt{2\pi}\tau_{e,\text{vel}}} \int_{-\infty}^{\infty} \exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_\eta^2}\right) \\
\times \left\{ s\left(r - n\mathrm{e}^{-(\zeta\omega_n+\omega_o)\tau_{e,\text{vel}}}\right) - s\left(r-n\right)\right\} dn,
\end{aligned} \tag{38}
$$

where $\tau_{e,\text{vel}} = -\frac{1}{\zeta\omega_n+\omega_o}\log\rho_c$.

**Acceleration** The total probability law for the response acceleration is

$$
f_{\ddot{x}}(r) = f_{\ddot{x}_b}(r)(1-\mathbb{P}_{r,\text{acc}}) + f_{\ddot{x}_r}(r)\mathbb{P}_{r,\text{acc}}, \tag{39}
$$

and this gives the following result for the acceleration pdf

$$
\begin{aligned}
f_{\ddot{x}}(r) = \frac{1}{\sigma_{\ddot{x}_b}\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma_{\ddot{x}_b}^2}\right)(1-\nu_\alpha \tau_{e,\text{acc}}) \\
+ \frac{\nu_\alpha \tau_{e,\text{acc}}}{r(\zeta\omega_n+\omega_o)\sigma_\eta\sqrt{2\pi}\tau_{e,\text{acc}}} \int_{-\infty}^{\infty} \exp\left(-\frac{(n-\mu_\alpha)^2}{2\sigma_\eta^2}\right) \\
\times \left\{ s\left(r - n(\zeta\omega_n+\omega_o)\mathrm{e}^{-(\zeta\omega_n+\omega_o)\tau_{e,\text{acc}}}\right) - s\left(r-n\left(\zeta\omega_n+\omega_o\right)\right)\right\} dn,
\end{aligned} \tag{40}
$$

where $\tau_{e,\text{acc}} = -\frac{1}{\zeta\omega_n+\omega_o}\log\rho_c$. Note that in this case we do not have the simple scaling as in the underdamped case for the conditionally rare pdf.

#### 3.3.1 Comparison with Monte-Carlo simulations

We confirm the accuracy of the analytical results given in equations (36), (38), and (40) for the strongly overdamped case through comparison with direct Monte-Carlo simulations. The parameters and resulted statistical quantities of the system are given in table 2. The analytical estimates show favorable agreement with numerical simulations for this case (figure 4), just as in the previous underdamped case.

Table 2: Parameters and relevant statistical quantities for SDOF system 2.

| | | | |
|---|---|---|---|
| $\lambda$ | 6 | $k$ | 1 |
| $T_\alpha$ | 1000 | $\zeta$ | 3 |
| $\omega_n$ | 1 | $\omega_d$ | 2.828 |
| $\mu_\alpha = 7\times\sigma_{\dot{h}}$ | 0.1 | $q$ | $1.582\times10^{-4}$ |
| $\sigma_\alpha = \sigma_{\dot{h}}$ | 0.0143 | $\sigma_h$ | 0.0063 |
| $\sigma_{\dot{x}_b}$ | 0.0056 | $\sigma_{x_b}$ | 0.0022 |
| $\sigma_\eta$ | 0.0154 | $\mathbb{P}_{r,dis}$ | 0.0140 |
| $\mathbb{P}_{r,vel}$ | 0.0004 | $\mathbb{P}_{r,acc}$ | 0.0004 |

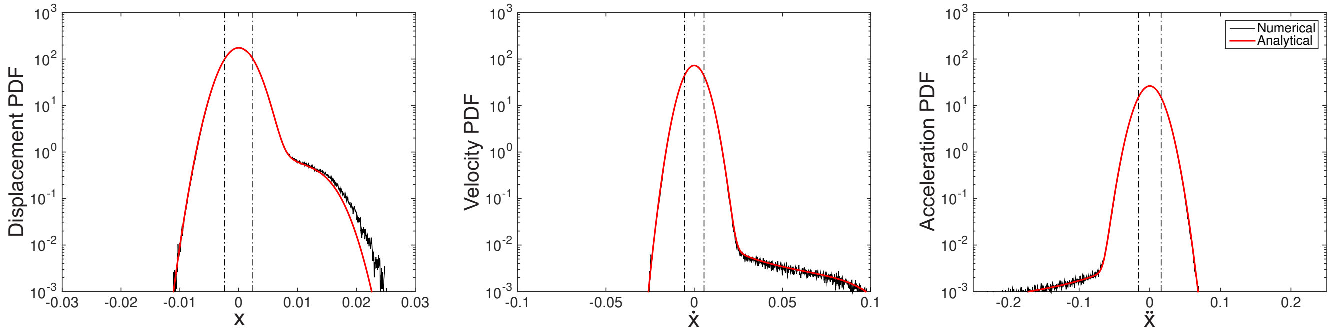


Figure 4: [**Severely overdamped case**] Comparison between direct Monte-Carlo simulation and the analytical pdf for SDOF system 2. The pdf for the value of each stochastic process is shown. The dashed line indicates one standard deviation.

# 4 Semi-analytical pdf of the response of SDOF systems

We now formulate a semi-analytical approach to quantify the response pdf for any arbitrary set of system parameters, including the severely underdamped or overdamped cases considered previously. The approach here adapts the numerical scheme described in [21] for systems undergoing internal instabilities.

While for the limiting cases that we studied previously knowledge of the trajectory (time series) of the system ($x_r(t)$ or $u_r(t)$) could analytically be translated to information about the corresponding pdf ($f_{x_r}$ or $f_{u_r}$), this is not always possible. In addition, for *nonlinear* structural systems one will not, in general, have analytical expressions for the rare event transitions. For these cases we can compute the rare event statistics by numerically approximating the corresponding histogram, using either analytical or numerically generated trajectories for the rare event regime.

## 4.1 Numerical computation of rare events statistics

Consider the same SDOF system introduced in section 3. Recall that we have quantified the response pdf by the PDS method using the total probability law

$$f_x(r) = f_{x_b}(r)(1 - \mathbb{P}_r) + f_{x_r}(r)\mathbb{P}_r. \tag{41}$$

In the previous section, the derivation consisted of estimating all three unknown quantities: the background distribution $f_{x_b}$, the rare event distribution $f_{x_r}$, and the rare event probability $\mathbb{P}_r$ analytically. However, in the semi-analytical scheme we will obtain the rare event distribution $f_{x_r}$ and rare event probability $\mathbb{P}_r$ by taking a histogram of the numerically simulated analytical form of the rare response. The background distribution $f_{x_b}$ is still obtained analytically as in section 3.1.

Recall that the rare event distribution is given by

$$f_{x_r}(r) = \int f_{x_r|\eta}(r \mid n) f_\eta(n)\, dn, \tag{42}$$

where $f_\eta(n)$ is known analytically (equation (18)). It is the conditional pdf $f_{x_r|\eta}(r \mid n)$ that we estimate by a histogram:

$$f_{x_r|\eta}(r \mid n) = \text{Hist}\big\{x_{r|\eta}(t \mid n)\big\}, \quad t = [0, \tau_{e,\text{dis}}], \tag{43}$$

where we use the analytical solution of the oscillator with non-zero initial velocity, $n$:

$$x_{r|\eta}(t \mid n) = \frac{n}{2\omega_o}\left(\mathrm{e}^{-(\zeta\omega_n-\omega_o)t} - \mathrm{e}^{-(\zeta\omega_n+\omega_o)t}\right). \tag{44}$$

The histogram is taken from $t = 0$ (the beginning of the rare event) until the end of the rare event at $t = \tau_e$. The conditional distribution of rare event response for the velocity and acceleration can be written as well:

$$f_{\dot{x}_r|\eta}(r \mid n) = \mathrm{Hist}\big\{\dot{x}_{r|\eta}(t \mid n)\big\}, \quad t = [0, \tau_{e,\mathrm{vel}}], \tag{45}$$

$$f_{\ddot{x}_r|\eta}(r \mid n) = \mathrm{Hist}\big\{\ddot{x}_{r|\eta}(t \mid n)\big\}, \quad t = [0, \tau_{e,\mathrm{acc}}]. \tag{46}$$

## 4.2 Numerical estimation of the rare events probability

In order to compute the histogram of a rare impulse event, the duration of a rare response needs to be obtained numerically. Recall that we have defined the duration of a rare responses by

$$x_r(\tau_e) = \rho_c \max\big\{|x_r|\big\}, \tag{47}$$

where $\rho_c = 0.1$. In the numerical computation of $\tau_e$ the absolute maximum of the response needs to be estimated numerically as well. Once the rare event duration has been specified, we can obtain the probability of rare events by

$$\mathbb{P}_r = \nu_\alpha \tau_e = \tau_e / T_\alpha. \tag{48}$$

This value is independent of the conditional background magnitude. The above procedure is applied for the rare event response displacement $\tau_{e,\mathrm{dis}}$, velocity $\tau_{e,\mathrm{vel}}$, and acceleration $\tau_{e,\mathrm{acc}}$.

## 4.3 Semi-analytical probability density functions

We can now compute the response pdf using the described semi-analytical approach. For the displacement we have:

$$f_x(r) = \frac{1-\nu_\alpha\tau_{e,\mathrm{dis}}}{\sigma_{x_b}\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma_{x_b}^2}\right) + \nu_\alpha\tau_{e,\mathrm{dis}} \int_0^\infty \mathrm{Hist}\big\{x_{r|\eta}(t \mid n)\big\} f_\eta(n)\, dn. \tag{49}$$

The corresponding pdf for the velocity $f_{\dot{x}}$ and acceleration $f_{\ddot{x}}$, can be computed with the same formula but with the appropriate variance for the Gaussian core ($\sigma_{\dot{x}_b}$ or $\sigma_{\ddot{x}_b}$), rare event duration ($\tau_{e,\mathrm{vel}}$ or $\tau_{e,\mathrm{acc}}$), and histograms ($\dot{x}_{r|\eta}(t \mid n)$ or $\ddot{x}_{r|\eta}(t \mid n)$).

### 4.3.1 Comparison with Monte-Carlo simulations

For illustration, a SDOF configuration is considered with critical damping ratio, $\zeta = 1$. The detailed parameters and relevant statistical quantities of the system are given in table 3. This is a regime where the analytical results derived in section 3 are not applicable. Even for this $\zeta$ value, the semi-analytical pdf for the response shows excellent agreement with direct simulations (figure 5). We emphasize that the computational cost of the semi-analytical scheme is comparable with that of the analytical approximations (order of seconds) and both are significantly lower than the cost of Monte-Carlo simulation (order of hours).

Table 3: Parameters and relevant statistical quantities for SDOF system 3.

| | | | |
|---|---|---|---|
| $\lambda$ | 2 | $k$ | 1 |
| $T_\alpha$ | 400 | $\zeta$ | 1 |
| $\omega_n$ | 1 | $\omega_d$ | 0 |
| $\mu_\alpha = 7 \times \sigma_{\dot{h}}$ | 0.1 | $q$ | $1.582 \times 10^{-4}$ |
| $\sigma_\alpha = \sigma_{\dot{h}}$ | 0.0143 | $\sigma_h$ | 0.0063 |
| $\sigma_{\dot{x}_b}$ | 0.0120 | $\sigma_{x_b}$ | 0.0052 |
| $\sigma_\alpha = \sigma_\eta$ | 0.0187 | $\mathbb{P}_{r,dis}$ | 0.0122 |
| $\mathbb{P}_{r,vel}$ | 0.0075 | $\mathbb{P}_{r,acc}$ | 0.0032 |

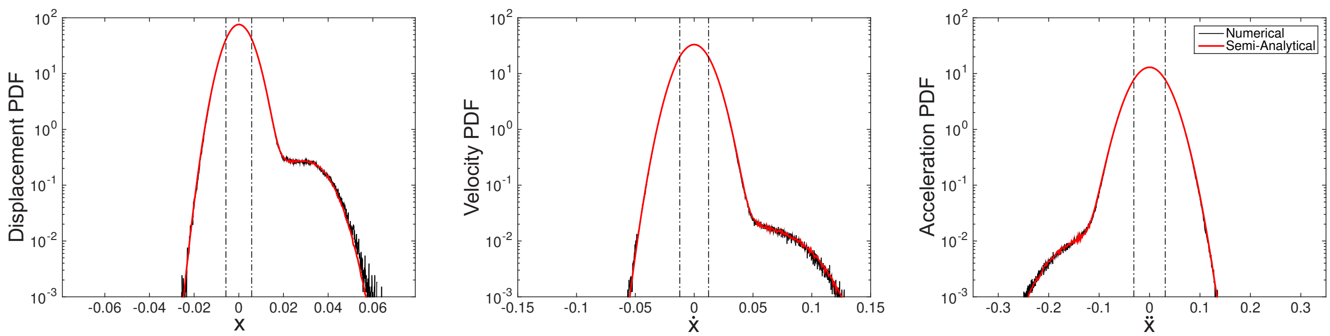


Figure 5: [**Critically damped system**] Comparison between direct Monte-Carlo simulations and the semi-analytical pdf for SDOF system 3. Dashed lines indicate one standard deviation.

# 5 Semi-analytical pdf for the response of MDOF systems

An important advantage of the semi-analytical scheme is the straightforward applicability of the algorithm to MDOF systems. In this section we demonstrate how the extension can be made for a two-degree-of-freedom (TDOF) linear system (see figure 6):

$$m\ddot{x} + \lambda\dot{x} + kx + \lambda_a(\dot{x} - \dot{y}) + k_a(x - y) = F(t), \tag{50}$$

$$m_a\ddot{y} + \lambda_a(\dot{y} - \dot{x}) + k_a(y - x) = 0, \tag{51}$$

where the stochastic forcing $F(t) = F_b(t) + F_r(t)$ is applied to the first mass (mass $m$) and $x, y$ are displacements of the two masses. As before, $F_b(t)$ is the background component and $F_r(t) = \sum_{i=1}^{N(t)} \alpha_i \, \delta(t - \tau_i)$ is the rare event component.

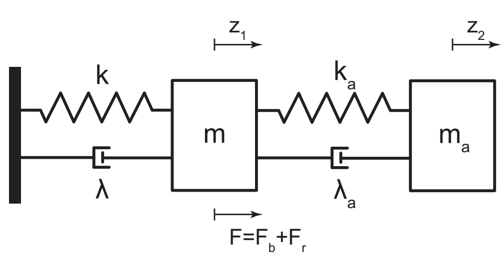


Figure 6: The considered TDOF system. The excitation is applied to the first mass, with mass $m$.

The background statistics are obtained by analyzing the response spectrum of the TDOF system subjected to the background excitation component. The details are given in Appendix C. For a nonlinear system this can be done using statistical linearization method [27]. The histograms for the rare event transitions can be computed through standard analytical expressions that one can derive for a linear system like the one we consider under the set of initial conditions: $x_0 = y_0 = \dot{y}_0 = 0$ and $\dot{x}_0 = n$.

Once the impulse response has been obtained we numerically quantify the rare event distribution, as well as the rare event duration and use the semi-analytical decomposition. The pdf is then given by

$$f_z(r) = \frac{1 - \nu_\alpha \tau_e^z}{\sigma_{z_b}\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma_{z_b}^2}\right) + \nu_\alpha \tau_e^z \int_0^\infty \text{Hist}\{z_{r|\eta}(t \mid n)\} f_\eta(n)\, dn, \tag{52}$$

where $z$ can be either of the degrees-of-freedom ($x$ or $y$) or the corresponding velocities or accelerations, while $\tau_e^z$ is the typical duration of the rare events and is estimated numerically. We note that as in the previous cases the pdf is composed of a Gaussian core (describing the background statistics) as well as, a heavy tailed component that is connected with the rare transitions. For each case of $z$ the corresponding variance under background excitation, temporal durations of rare events, and histograms for rare events should be employed.

Results are presented for the pdf of the displacement, velocity and acceleration of each degree-of-freedom (figure 7). These compare favorably with the direct Monte-Carlo simulations. The parameters and resulted statistical quantities of the system are given in table 4. Further numerical simulations (not presented) demonstrated strong robustness of the approach.

Table 4: Parameters and relevant statistical quantities for the TDOF system.

| | | | |
|---|---|---|---|
| $m$ | 1 | $m_a$ | 1 |
| $\lambda$ | 0.01 | $k$ | 1 |
| $\lambda_a$ | 1 | $k_a$ | 0.1 |
| $T_\alpha$ | 1000 | $\sigma_\eta$ | 0.0199 |
| $\mu_\alpha$ | 0.1 | $q$ | $1.582 \times 10^{-4}$ |
| $\sigma_\alpha$ | 0.0143 | $\sigma_{F_b}$ | 0.0351 |
| $\mathbb{P}^x_{r,dis}$ | 0.0177 | $\mathbb{P}^y_{r,dis}$ | 0.0190 |
| $\mathbb{P}^x_{r,vel}$ | 0.0098 | $\mathbb{P}^y_{r,vel}$ | 0.0209 |
| $\mathbb{P}^x_{r,acc}$ | 0.0066 | $\mathbb{P}^y_{r,acc}$ | 0.0082 |

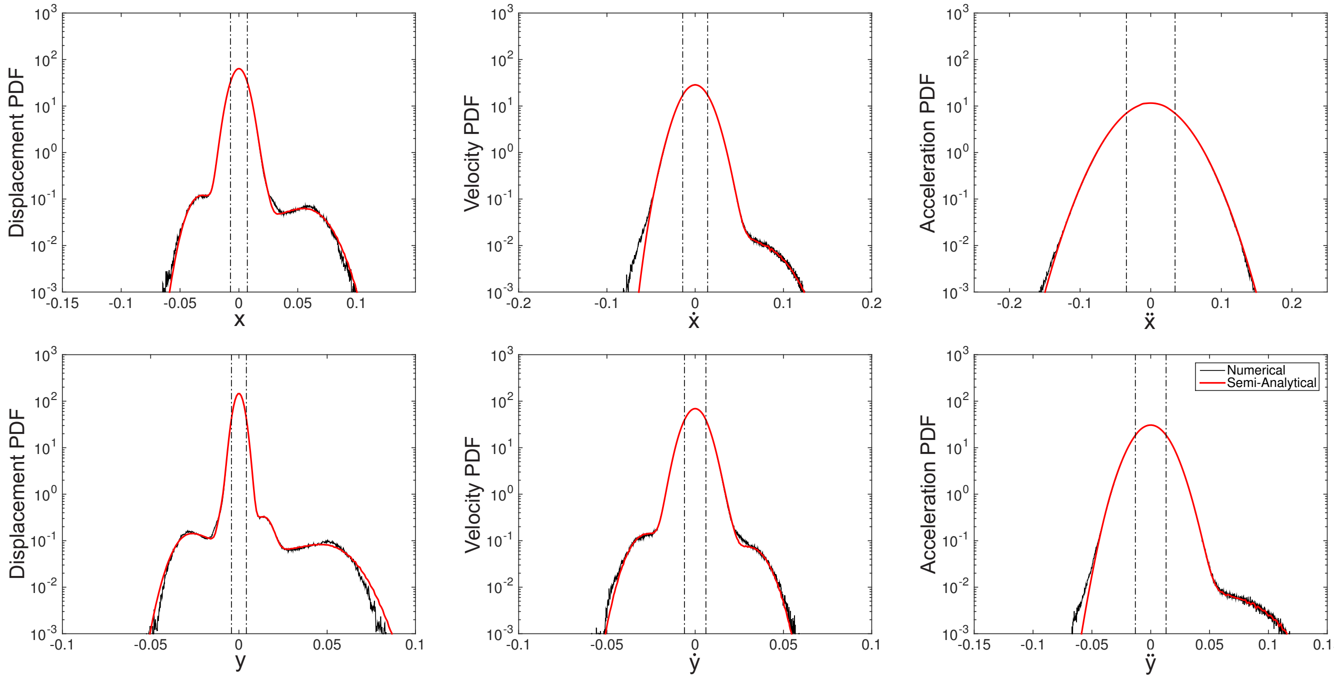


Figure 7: **[Two DOF System]** Comparison between direct Monte-Carlo simulation and the semi-analytical approximation. The pdf for the value of the time series are presented. Dashed line indicates one standard deviation.

# 6 Semi-analytical pdf of local maxima

It is straightforward to extend the semi-analytical framework for other quantities of interest, such as the local extrema (maxima and minima) of the response. The numerically generated histogram for the rare transitions can be directly computed for the local extrema as well. On the other hand, for the background excitation regime, we use known results from the theory of stationary Gaussian stochastic processes to describe the corresponding background pdf analytically.

## 6.1 Distribution of local maxima under background excitation

For a stationary Gaussian process with arbitrary spectral bandwidth $\epsilon$, the probability density function of positive extrema (maxima) is given by [24, 13]:

$$f_{m^+}(\xi) = \frac{\epsilon}{\sqrt{2\pi}} e^{-\xi^2/2\epsilon^2} + \sqrt{1-\epsilon^2}\xi e^{-\xi^2/2}\Phi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}\xi\right), \quad -\infty \leq \xi \leq \infty, \tag{53}$$

where $\xi = \frac{x}{\sqrt{\mu_0}}$, $x$ is the magnitude of the maxima, spectral bandwidth $\epsilon = \sqrt{1 - \frac{\mu_2^2}{\mu_0\mu_4}}$, and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$ is the standard normal cumulative distribution function. The spectral moments for the background response displacement $x_b$ are also defined as

$$\mu_n = \int_0^\infty \omega^n S_{x_b x_b}(\omega)\, d\omega. \tag{54}$$

We note that that for the limit of an infinitesimal narrow-banded signal ($\epsilon = 0$), the pdf converges to a Rayleigh distribution. On the other hand for an infinitely broad-banded signal ($\epsilon = 1$), the distribution converges to the Gaussian pdf. For a signal with in-between spectral bandwidth ($0 \le \epsilon \le 1$), the pdf has a blended structure with the form in equation (53).

Taking into account the asymmetric nature of the intermittent excitation, we need to consider both the positive and negative extrema. For the background excitation the pdf can be written as:

$$f_{\hat{x}_b}(x) = \frac{1}{2\sqrt{\mu_0}} \Big\{ f_{m^+}\Big(\frac{x}{\mu_0}\Big) + f_{m^+}\Big(-\frac{x}{\mu_0}\Big) \Big\}, \quad -\infty \le x \le \infty, \tag{55}$$

where the $\hat{x}$ notation denotes the local extrema of $x$. Similar expressions can be obtained for the velocity and acceleration extrema.

## 6.2 Statistics of local extrema during rare transitions

The conditional pdf $f_{\hat{x}_r|\eta}$ for the local extrema can be numerically estimated through the histogram:

$$f_{\hat{x}_r|\eta}(r \mid n) = \mathrm{Hist}\big\{ \mathcal{M}\left( x_{r|\eta}(t \mid n) \right) \big\}, \quad t = [0, \tau_{e,\mathrm{dis}}], \tag{56}$$

where $\mathcal{M}(\,\cdot\,)$ is an operator that gives all the positive/negative extrema. The positive/negative extrema are defined as the points where the derivative of the signal is zero.

## 6.3 Semi-analytical probability density function for local extrema

The last step consists of applying the decomposition of the pdf. This takes the form:

$$f_{\hat{x}}(r) = (1 - \nu_\alpha \tau_{e,\mathrm{dis}})\, f_{\hat{x}_b}(r) + \nu_\alpha \tau_{e,\mathrm{dis}} \int_0^\infty \mathrm{Hist}\big\{ \mathcal{M}\left( x_{r|\eta}(t \mid n) \right) \big\} f_\eta(n)\, dn. \tag{57}$$

The same decomposition can be used for the velocity and acceleration local maxima. We compare the semi-analytical decomposition with Monte-Carlo simulations. In figure 8 we present results for the two-degree-of-freedom system. The pdf are shown for local extrema of the displacements, velocities and accelerations for each degree of freedom. We emphasize the non-trivial structure of the pdf and especially their tails. Throughout these comparisons the semi-analytical scheme demonstrates accurate estimation of both the heavy tails and the non-Gaussian/non-Rayleigh structure of the background local extrema distribution.

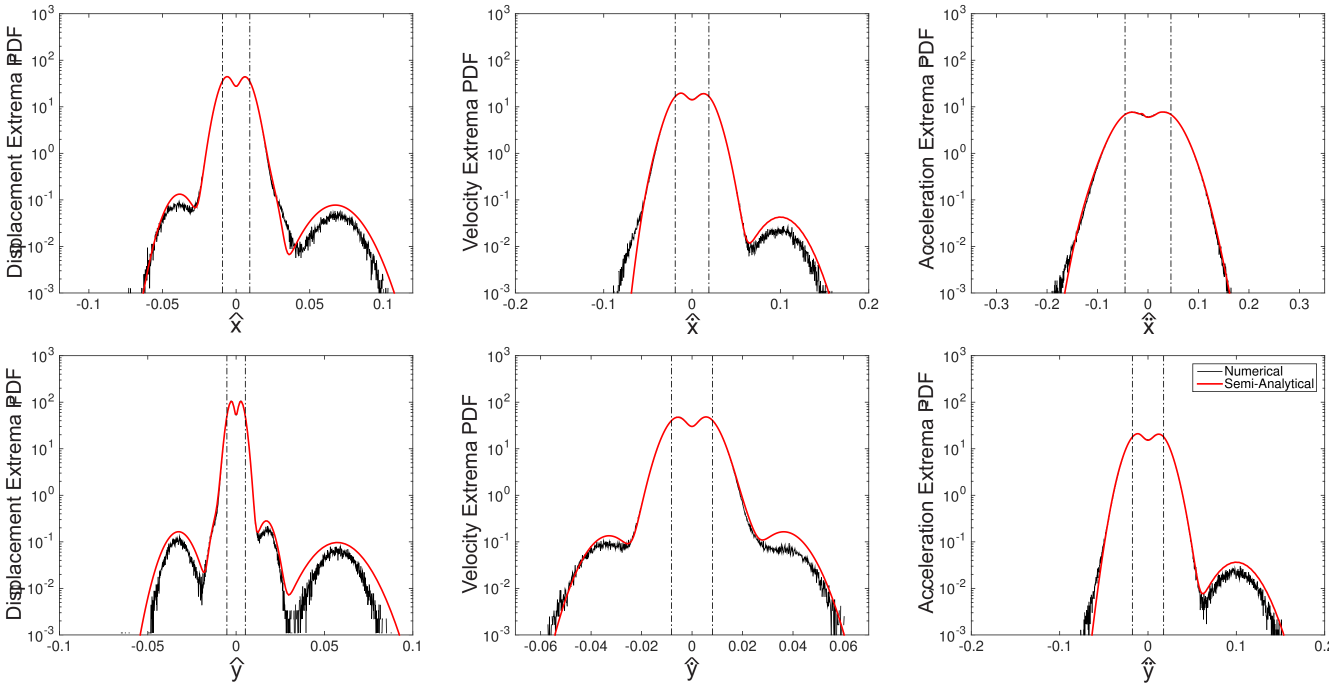


Figure 8: [**Local extrema for TDOF**] Comparison between direct Monte-Carlo simulation and the semi-analytical approximation. The pdf for the local extrema of the response are presented. Dashed line indicates one standard deviation.

# 7 Summary and conclusions

We have formulated a robust approximation method to quantify the probabilistic response of structural systems subjected to stochastic excitation containing intermittent components. The foundation of our approach is the recently developed probabilistic decomposition-synthesis method for the quantification of rare events due to internal instabilities to the problem where extreme responses are triggered by external forcing. The intermittent forcing is represented as a background component, modeled through a colored processes with energy distributed across a range of frequencies, and additionally a rare/extreme component that can be represented by impulses that are Poisson distributed with large inter-arrival time. Owing to the nature of the forcing, even the probabilistic response of a linear system can be highly complex with asymmetry and complicated tail behavior that is far from Gaussian, which is the expected form of the response pdf if the forcing did not contain an intermittently extreme component.

The main result of this work is the derivation of analytical/semi-analytical expressions for the pdf of the response and its local extrema for structural systems (including the response displacement, velocity, and acceleration pdf). These expressions decompose the pdf into a probabilistic core, capturing the statistics under background excitation, as well as a heavy-tailed component associated with the extreme transitions due to the rare impacts. We have performed a thorough analysis for linear SDOF systems under various system parameters and also derived analytical formulas for two special cases of parameters (lightly damped or heavily damped systems). The general semi-analytical decomposition is applicable for any arbitrary set of system parameters and we have demonstrated its validity through comprehensive comparisons with Monte- Carlo simulations. The general framework is also directly applicable to multi-degree-of-freedom MDOF systems, as well as systems with nonlinearities, and we have assessed its performance through a 2DOF linear system of two coupled oscillators excited through the first mass. Modifications of the method to compute statistics of local extrema have also been presented.

The developed approach allows for computation of the response pdf of structural systems many orders of magnitude faster than a direct Monte-Carlo simulation, which is currently the only reliable tool for such computations. The rapid evaluation of response pdfs for systems excited by extreme forcing by the method presented in this work paves the way for enabling robust design of structural systems subjected to extreme events of a stochastic nature. Our future endeavors include the application of the developed framework to the optimization of engineering systems where extreme event mitigation is required. In such cases, it is usually not feasible to run Monte-Carlo simulations for various parameter sets during the design process owing to the computational costs associated with low probability rare events, let alone perform parametric optimization. We believe our approach is well suited to such problems and can prove to be an important method for engineering design and reliability assessment.

# Acknowledgments

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# A Impulse response of SDOF systems

The response of the system

$$\ddot{x}_r(t) + \lambda \dot{x}_r(t) + k x_r(t) = 0, \tag{58}$$

under an impulse $\alpha$ at an arbitrary time $t_0$, say $t_0 = 0$, and with zero initial value but nonzero initial velocity ($(x_r, \dot{x}_r) = (0, \dot{x}_{r0})$ at $t = 0^-$) are given by the following equations under the two limiting cases of damping:

**Severely underdamped case $\zeta \ll 1$** With the approximation of $\zeta \ll 1$ (or $\omega_d \approx \omega_n$), we can simplify responses as

$$x_r(t) = \frac{\alpha + \dot{x}_{r0}}{\omega_d} \mathrm{e}^{-\zeta\omega_n t} \sin \omega_d t, \tag{59}$$

$$\dot{x}_r(t) = (\alpha + \dot{x}_{r0}) \mathrm{e}^{-\zeta\omega_n t} \cos \omega_d t, \tag{60}$$

$$\ddot{x}_r(t) = -(\alpha + \dot{x}_{r0}) \omega_d \mathrm{e}^{-\zeta\omega_n t} \sin \omega_d t. \tag{61}$$

**Severely overdamped case $\zeta \gg 1$** Similarly, with the approximation of $\zeta \gg 1$ (or $\omega_o \approx \zeta\omega_n$), we can simplify responses as

$$x_r(t) = \frac{(\alpha + \dot{x}_{r0})}{2\omega_o} \mathrm{e}^{-(\zeta\omega_n - \omega_o)t}, \tag{62}$$

$$\dot{x}_r(t) = (\alpha + \dot{x}_{r0}) \mathrm{e}^{-(\zeta\omega_n + \omega_o)t}, \tag{63}$$

$$\ddot{x}_r(t) = -(\zeta\omega_n + \omega_o)(\alpha + \dot{x}_{r0}) \mathrm{e}^{-(\zeta\omega_n + \omega_o)t}. \tag{64}$$

## B Probability distribution for an arbitrarily exponentially decaying function

Consider an arbitrary time series in the following form:

$$x(t) = \mathcal{A}\mathrm{e}^{-\alpha t}, \quad \text{where } t \sim \text{Uniform}\,(\tau_1, \tau_2)\,, \tag{65}$$

where $\mathcal{A}, \alpha > 0$ and $\tau_1 < \tau_2$. The cumulative distribution function (cdf) of $x(t)$ is

$$\begin{aligned}
F_x(x) &= \mathbb{P}(\mathcal{A}\mathrm{e}^{-\alpha t} < x), && (66)\\
&= \mathbb{P}\Big(t > \frac{1}{\alpha}\log(\mathcal{A}/x)\Big), && (67)\\
&= 1 - \mathbb{P}\Big(t < \frac{1}{\alpha}\log(\mathcal{A}/x)\Big), && (68)\\
&= 1 - \int_{-\infty}^{\frac{1}{\alpha}\log(\mathcal{A}/x)} f_T(t)\,dt, && (69)
\end{aligned}$$

where $f_T(t)$ is expressed using the step function $s(\,\cdot\,)$ as:

$$f_T(t) = \frac{1}{\tau_2 - \tau_1}\big\{s(t-\tau_1) - s(t-\tau_2)\big\}, \quad \tau_1 < \tau_2. \tag{70}$$

The pdf of the response $x(t)$ can then be derived by differentiation.

$$\begin{aligned}
f_x(x) =& \frac{d}{dx}F_x(x), && (71)\\
=& \frac{1}{\alpha x} f_T\Big(\frac{1}{\alpha}\log(\mathcal{A}x)\Big), && (72)\\
=& \frac{1}{\alpha x(\tau_2 - \tau_1)}\Big\{s(x - \mathcal{A}\mathrm{e}^{-\alpha\tau_2}) - s(x - \mathcal{A}\mathrm{e}^{-\alpha\tau_1})\Big\}. && (73)
\end{aligned}$$

We utilize the above formula for deriving analytical pdfs. Note that the step function with respect to $x$ in the above has been derived using

$$\begin{aligned}
\tau_1 < t < \tau_2, && (74)\\
-\alpha\tau_2 < -\alpha t < -\alpha\tau_1, && (75)\\
\mathcal{A}\mathrm{e}^{-\alpha\tau_2} < x < \mathcal{A}\mathrm{e}^{-\alpha\tau_1}. && (76)
\end{aligned}$$

## C Background response for TDOF system

Consider the statistical response of the system to the background forcing component,

$$\begin{aligned}
m\ddot{x}_b + \lambda\dot{x}_b + kx_b + \lambda_a(\dot{x}_b - \dot{y}_b) + k_a(x_b - y_b) = F_b(t), && (77)\\
m_a\ddot{y}_b + \lambda_a(\dot{y}_b - \dot{x}_b) + k_a(y_b - x_b) = 0.
\end{aligned}$$

The spectral densities are given by

$$S_{x_b x_b}(\omega) = \frac{\omega^4 S_{F_b}(\omega)}{\left\{\mathcal{A}(\omega) - \frac{\mathcal{B}(\omega)^2}{\mathcal{C}(\omega)}\right\}\left\{\mathcal{A}(-\omega) - \frac{\mathcal{B}(-\omega)^2}{\mathcal{C}(-\omega)}\right\}}, \tag{78}$$

$$S_{\dot{x}_b \dot{x}_b}(\omega) = \omega^2 S_{x_b x_b}(\omega), \tag{79}$$

$$S_{\ddot{x}_b \ddot{x}_b}(\omega) = \omega^4 S_{x_b x_b}(\omega), \tag{80}$$

$$S_{y_b y_b}(\omega) = \frac{\omega^4 S_{F_b}(\omega)}{\left\{\frac{\mathcal{A}(\omega)\mathcal{C}(\omega)}{\mathcal{B}(\omega)} - \mathcal{B}(\omega)\right\}\left\{\frac{\mathcal{A}(-\omega)\mathcal{C}(-\omega)}{\mathcal{B}(-\omega)} - \mathcal{B}(-\omega)\right\}}, \tag{81}$$

$$S_{\dot{y}_b \dot{y}_b}(\omega) = \omega^2 S_{y_b y_b}(\omega), \tag{82}$$

$$S_{\ddot{y}_b \ddot{y}_b}(\omega) = \omega^4 S_{y_b y_b}(\omega), \tag{83}$$

where $S_{F_b}(\omega)$ is the spectral density of $F_b(t)$, and

$$\mathcal{A}(\omega) = (\lambda_a + \lambda)(j\omega) + (k_a + k) - m\omega^2, \tag{84}$$

$$\mathcal{B}(\omega) = \lambda_a(j\omega) + k_a, \tag{85}$$

$$\mathcal{C}(\omega) = \lambda_a(j\omega) + k_a - m_a\omega^2. \tag{86}$$

Thus, we can obtain the following conditionally background variances

$$\sigma_{x_b}^2 = \int_0^\infty S_{x_b x_b}(\omega)\, d\omega, \quad \sigma_{\dot{x}_b}^2 = \int_0^\infty S_{\dot{x}_b \dot{x}_b}(\omega)\, d\omega, \quad \sigma_{\ddot{x}_b}^2 = \int_0^\infty S_{\ddot{x}_b \ddot{x}_b}(\omega)\, d\omega, \tag{87}$$

$$\sigma_{y_b}^2 = \int_0^\infty S_{v_b v_b}(\omega)\, d\omega, \quad \sigma_{\dot{y}_b}^2 = \int_0^\infty S_{\dot{y}_b \dot{y}_b}(\omega)\, d\omega, \quad \sigma_{\ddot{y}_b}^2 = \int_0^\infty S_{\ddot{y}_b \ddot{y}_b}(\omega)\, d\omega. \tag{88}$$

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